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Robust economic model predictive control for periodic operation

With application to supply chain networks

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Abstract

We consider economic control of systems, which are optimally operated at some periodic orbit. In particular, our motivation is to economically control supply chain networks. First we provide a detailed analysis of optimal periodic operation in case of linear periodic time varying systems with convex cost functionals. In case of piece-wise linear cost functionals we derive an explicit linear programming formulation in order to verify optimal closed loop operation at a given periodic orbit as well as suboptimal operation for any other, feasible system trajectory off that very orbit. Furthermore, we present a novel economic model predictive control scheme for general non-linear systems based on a terminal cost and a terminal constraint set. Besides recursive feasibility and asymptotic stability of the control scheme, we strictly proof an asymptotic average performance which is not worse than the performance value of the systems optimal periodic orbit. Using a tube-based approach, we extend our method to become applicable in the presence of unknown but bounded disturbances. In addition, we propose the concept of robust optimal periodic operation which turns out to essentially improve the closed loop performance for the supply chain example considered, under the presence of disturbances. Throughout this work, we illustrate each new concept using a simple supply chain model. Lastly, we perform an in-depth experimental analysis of a more complex supply chain network consisting of a supplier, a transportation network and three retail stores. We compare our nominal and robust control schemes with an existing, terminal cost and terminal set free method.

Zusammenfassung

Wir betrachten die ökonomische Regelung von Systemen, die optimal periodisch betrieben werden. Die Motivation der vorliegenden Arbeit liegt insbesondere in der ökonomischen Regelung von Versorgungsnetzwerken. Zuerst analysieren wir optimales, periodisches Verhalten im Fall von linearen periodischen zeitvarianten Systemen mit konvexer Kostenfunktion. Im Falle von stückweise definierter linearer Kostenfunktionale stellen wir ein explizites, lineares Optimierungsproblem auf, mit welchem optimales Systemverhalten an einem speziellen periodischen Systemorbit gezeigt werden kann. Des Weiteren kann damit suboptimales Systemverhalten entlang aller anderen, möglichen Systemorbits verifiziert werden. Darüber hinaus stellen wir ein neuartiges ökonomisches modellprädiktives Regelungsverfahren vor, welches auf generelle nichtlineare Systeme angewendet werden kann und auf Endkosten und einer Endregion basiert. Neben rekursiver Lösbarkeit und asymptotischer Stabilität der Regelungsmethode beweisen wir, dass die asymptotische durchschnittliche Leistung nicht schlechter als die des optimalen periodischen Systemorbits ist. Unter Verwendung eines "röhrenbasierten" Ansatzes erweitern wir unsere Regelungsmethode dahingehend, dass diese auch im Fall von unbekannten begrenzten Störungen anwendbar ist. In diesem Zuge führen wir das Konzept des robusten optimalen periodischen Orbits ein, um die Leistung des geschlossenen Regelkreises im Fall von Störungen zu verbessern. Durchgehend wird jedes neue Konzept welches wir vorstellen anhand eines einfachen Versorgungsnetzwerkes veranschaulicht. Abschließend wird ein komplexeres Versorgungsnetzwerk analysiert, welches aus einem Zulieferer, einem Transportationsnetzwerk und drei Einzelhändlern besteht. Wir vergleichen unsere nominellen und robusten Regelungsverfahren mit einer existierenden Methode ohne Endkosten und Endregion.

Contents

1 Introduction

This work consists of four parts. Each part begins with a respective introduction which also links our results to existing literature. We use a simple supply chain example for illustrating each new concept on a common basis, before presenting the actual application in the last part.

Optimal operation | Chapter 2
How should the system be operated? Which periodic orbit is optimal? Is it possible to develop an algorithm for verifying (unique) optimal operation in a linear time varying setting?

Economic model predictive control for optimal periodic operation using terminal constraints | Chapter 3
How to design an online optimization problem in order to guarantee the best, periodic system performance and convergence to the best periodic orbit?

Robust economic model predictive control for robust optimal periodic operation | Chapter 4
How to improve the performance and ensure safe periodic system operation under the presence of disturbances?

Application to a supply chain network | Chapter 5
How to apply the theoretical results using the example of a supply chain network?

1.1 Notation and basic definitions

In the following we introduce common notations and definitions, repetitively used.

Set of integers in the interval $[a,b] \subset \mathbb{R}$

$$\mathcal{I}_{[a,b]}. \tag{1.1}$$

Set of integers in the interval greater than or equal to a

$$\mathcal{I}_{\geq a}. \tag{1.2}$$

The distance between $x \in \mathbb{R}^n$ and a set $\mathcal{W} \subseteq \mathbb{R}$

$$|x|_{\mathcal{W}} := \inf_{a \in \mathcal{W}} |x - a|. \tag{1.3}$$

For a set $\mathcal{A} \subseteq \mathbb{R}^n$ and $\epsilon > 0$, define the neighbourhood

$$\mathcal{B}_\epsilon(\mathcal{A}) := \{x \in \mathbb{R}^n : |x|_{\mathcal{A}} \leq \epsilon\}. \tag{1.4}$$

Definition 1.1.1 (Class $\mathcal{K}$ functions)**.** A continuous function $\alpha : [0,a) \to [0,\infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if it belongs to class $\mathcal{K}$, $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$. ■

Definition 1.1.2 (Class $\mathcal{L}$ functions)**.** A continuous function $\sigma : [0,\infty) \to (0,\infty)$ is said to belong to class $\mathcal{L}$ if it is strictly decreasing and $\lim_{r\to\infty} \sigma(r) = 0$. ■

Definition 1.1.3 (Class $\mathcal{KL}$ functions)**.** A continuous function $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ is said to belong to class $\mathcal{KL}$ if it is class $\mathcal{K}$ in its first argument and class $\mathcal{L}$ in its second argument. ■

1.2 Preliminaries

Consider systems of the type

$$x(k+1) = f(x(k),\ u(k), w(k)), x(0) = x, k \in \mathcal{I}_{\geq 0} \tag{1.5}$$

with $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ subject to pointwise-in-time state and input constraints[1] $x(k) \in \mathbb{X} \subseteq \mathbb{R}^n$ and $u(k) \in \mathbb{U} \subseteq \mathbb{R}^m$. And unkown but bounded disturbances $w(k) \in \mathbb{W} \subset \mathbb{R}^q$. Given a control sequence $u = \{u(0),...,u(K)\} \in \mathbb{U}^{K+1}$ ($K \in \mathcal{I}_{\geq 0}$ can attain ∞) we denote the corresponding solution of (1.5) by $x_u = \{x_u(0,x),..., x_u(K+1,x)\} \in \mathbb{X}^{K+2}$ with initial condition $x_u(0,x) = x$. Consider a constant disturbance $\mathbb{W} = \{0\}$ for all $k \in \mathcal{I}_{\geq 0}$. Then for a given $x \in \mathbb{X}$ the set of all feasible control sequences of length $T \in \mathcal{I}_{\geq 0}$, denoted by $\mathbb{U}^T(x)$ (T can attain ∞), satisfies $u(k) \in \mathbb{U}$ for all $k \in \mathcal{I}_{[0,T-1]}$ and $x_u(k,x) \in \mathbb{X}$ for all $k \in \mathcal{I}_{[0,T]}$.

System (1.5) is endowed with a stage cost function

$$\ell : \mathbb{X} \times \mathbb{U} \to \mathbb{R} \tag{1.6}$$

that is assumed to be bounded from below on $\mathbb{X} \times \mathbb{U}$. Without loss of generality we assume that $0 \leq \inf_{x\in\mathbb{X},u\in\mathbb{U}} \ell(x,u)$.

Definition 1.2.1 (Nominal feasible P-periodic orbits [19]). Consider $\mathbb{W} = \{0\}$. A set of state and input pairs $\Pi = \{(x_0^p,u_0^p),...,(x_{P-1}^p, u_{P-1}^p)\}$ with $P \in \mathcal{I}_{\geq 1}$ is called a nominal feasible P-periodic orbit of system (1.5) if $x_k^p \in \mathbb{X}$, $u_k^p \in \mathbb{U}$, and $x_{k+1}^p = f(x_k^p,u_k^p,0)$ for all $k \in \mathcal{I}_{[0,P-2]}$ and $x_0^p = f(x_{P-1}^p,u_{P-1}^p,0)$. It is called a minimal P-periodic orbit if $x_{k1}^p \neq x_{k2}^p$ with $k1 \neq k2$. The projection of Π on $\mathbb{X}$ is denoted by $\Pi_{\mathbb{X}} := \{x_0^p,...,x_{P-1}^p\}$ and the projection of Π on $\mathbb{U}$ by $\Pi_{\mathbb{U}} := \{u_0^p,...,u_{P-1}^p\}$ respectively. S_Π^P denotes the set of all nominal feasible P-periodic orbits. ■

Remark 1.2.2 (Identification of the optimal orbit). *The minimal nominal optimal P-periodic orbit is obtained by solving*

$$\{P^*, \Pi^*\} = \underset{P\in\mathcal{I}_{\geq 1},\Pi\in S_\Pi^P}{\operatorname{argmin}} \frac{1}{P}\sum_{k=0}^{P-1} \ell(x_k^p,u_k^p) \tag{1.7}$$

and choosing the solution pair with the smallest value for P in case of non-uniqueness. The cost $\sum_{k=0}^{P-1} \ell(x_k^p,u_k^p)$ will be denoted by $\tilde{\ell}_\Pi$. ■

Definition 1.2.3 (P-step system [11]). The (general) P-step system of system (1.5) is defined with states $\tilde{x} := (x_0,...,x_{P-1}) \in \mathbb{X}^P$, inputs $\tilde{u} := (u_0,...,u_{P-1}) \in \mathbb{U}^P$, disturbances $\tilde{w} := (w_0,...,w_{P-1}) \in \mathbb{W}^P$ and dynamics

$$f^P(\tilde{x}, \tilde{u}, \tilde{w}) := (f(x_{P-1},u_0,w_0), f(f(x_{P-1},u_0,w_0),u_1,w_1), ...) \tag{1.8}$$

[1] All the results presented in this work can easily be extended to coupled state and input constraints. We consider the decoupled case for easier readability.

which defines[2]

$$\tilde{x}^+ = f^P(\tilde{x}, \tilde{u}, \tilde{w}). \tag{1.9}$$

The initial condition for a solution of system (1.9) is $x_{P-1}(0) = x \in \mathbb{X}$ (initial conditions of other states are not relevant for the solution). Given an initial condition x, a control and disturbance sequence $\boldsymbol{u} \in \mathbb{U}^{PK}$ and $\boldsymbol{w} \in \mathbb{W}^{PK}$, $K \in \mathcal{I}_{\geq 1}$, the corresponding solution is denoted by[3]

$$\tilde{x}_{\boldsymbol{u}}(k, x) = \begin{cases} (0,..,0, x_{\boldsymbol{u}}(k-P+i, x), ..., x_{\boldsymbol{u}}(k, x)), \\ \qquad\qquad \text{for } k-P+i = 0, i \in \mathcal{I}_{[0,k-1]}, \\ (x_{\boldsymbol{u}}(k-P+1, x), x_{\boldsymbol{u}}(k-P+2, x), .., x_{\boldsymbol{u}}(k, x), \\ \qquad\qquad \text{else.} \end{cases}$$

Further, the stage cost function for the P-step system is given by

$$\tilde{\ell}(\tilde{x}, \tilde{u}) := \sum_{i=0}^{P-1} \ell(x_{\tilde{u}}(i, x_{P-1}), u_i) \tag{1.10}$$

which possibly corresponds to the cost along a P-periodic orbit. However, in the non-periodic case, the cost corresponds to the sum of P-stage costs along the original system (1.5), when starting at x_{P-1} and applying the inputs $u_i, i = 0,..,P-1$. As a measure of distance between a state and input pair of the P-step system $\{\tilde{x}, \tilde{u}\}$ and a nominal P-periodic orbit Π of system (1.5) we define via the distance measure (1.3)

$$|(\tilde{x}, \tilde{u})|_{\Pi} := \sum_{i=0}^{P-1} |(x_{\tilde{u}}(i, x_{P-1}), u_i)|_{\Pi} \tag{1.11}$$

and

$$|\tilde{x}|_{\Pi_{\mathbb{X}}} := \sum_{i=0}^{P-1} |x_{\tilde{u}}(i, x_{P-1})|_{\Pi_{\mathbb{X}}}. \tag{1.12}$$

■

[2] Given a current state $\tilde{x}$, the expression $\tilde{x}^+$ indicates the successor, $\tilde{x}^-$ the predecessor P-step state and $\tilde{u}^+$ the sucessor, and $\tilde{u}^-$ the predecessor inputs.

[3] The definition implies $\tilde{x}_{\boldsymbol{u}}(k+1, x) \neq f^P(\tilde{x}_{\boldsymbol{u}}(k, x), \tilde{u}, 0)$ and $\tilde{x}_{\boldsymbol{u}}(k+P, x) = f^P(\tilde{x}_{\boldsymbol{u}}(k, x), \tilde{u}, 0)$.

Remark 1.2.4. *Note that any feasible nominal periodic orbit $\Pi \in S_{\Pi}^{P}$ of system* (1.5)*, is an equilibrium point of the P-step system. I.e.*

$$f^{P}\left((x_0^p, ..., x_{P-1}^p), (u_0^p, ..., u_{P-1}^p), 0\right) = (x_0^p, ..., x_{P-1}^p).$$

■

In the following we assume that S_{Π}^{P} is non-empty and that a (possibly non-unique) optimal trajectory exists that satisfies (1.7).

Definition 1.2.5 (Rotated P-step states and inputs)**.** Consider a periodic orbit Π and let

$$\tilde{\Pi} := \left\{\left((x_k^p, u_k^p), .., (x_{P-1}^p, u_{P-1}^p), .., (x_{k-1}^p, u_{k-1}^p)\right) : k \in \mathcal{I}_{[0,P-1]}\right\}$$

be the set of phase shifted periodic orbits according to Π. We denote the corresponding projections on the states and inputs by $\tilde{\Pi}_{\mathbb{X}}$ and $\tilde{\Pi}_{\mathbb{U}}$. ■

1.3 Central example: Simple supply chain network

Each new concept and method presented in this work will be illustrated using the example of (economically) controlling a simple supply chain network. It is designed to be as simple as possible while still containing all the characteristics considered in this work. In particular this includes a graph and real valued system state structure, additive disturbances and an optimal operating behavior that is periodic. The relevance of the considered model for the field of supply chain networks is discussed in Sec. 5.1. The simple supply chain network consists of one supplier, one retailer, and one truck for transportation of the goods, see Fig 1.1.

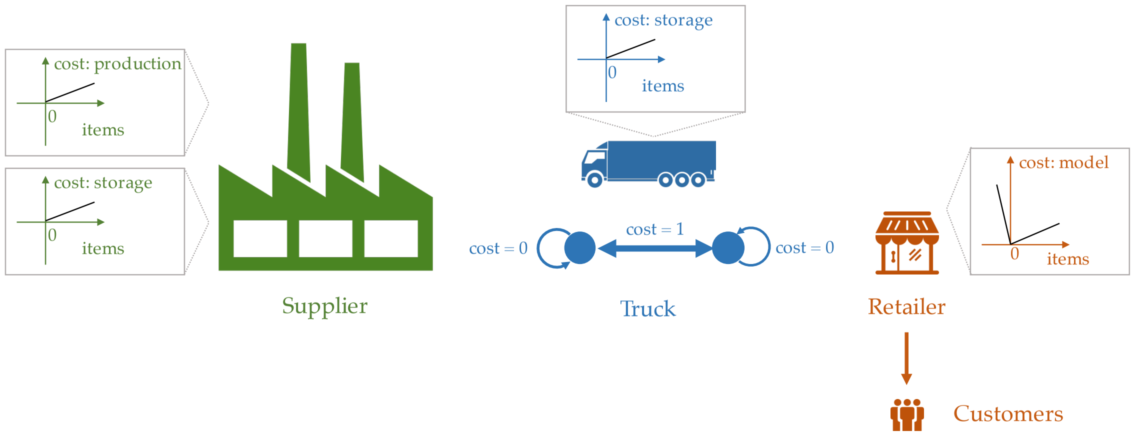


Figure 1.1: Elements of the simple supply chain network example. Illustrated are the qualitative cost functions and the graph on which the truck is able to drive.

Dynamics

We use the following notations: $x_{S,1}(k) \in \mathbb{R}$ represents the number of goods in the supplier production process, $x_{S,2}(k) \in \mathbb{R}$ the number of goods in the supplier storage, $x_{T,P}(k) \in \{0,1\}$ describes the truck position, $x_{T,L}(k) \in \mathbb{R}$ the number of goods which are carried by the truck and $x_R(k) \in \mathbb{R}$ the number of goods in the retailer's storage. Inputs are represented using the truck navigation $u_{T,P} \in \{0,1\}$, the truck loading of goods $u_{T,L} \in \mathbb{R}$, and the supplier production request (number of goods) $u_S \in \mathbb{R}$ as well as external disturbances (number of goods) $w \in \mathbb{W}$ where $w(k) = w^* + \epsilon$ with $\epsilon \sim Y_\epsilon$ with probability distribution $P(\epsilon)$ that describes the costumer demand at the retail store. We assume $\mathbb{E}[\epsilon] = 0$ from which follows that we have $\mathbb{E}[w(k)] = w^*$. The corresponding switched system dynamics is given as

$$\begin{bmatrix} x_{S,1}(k+1) \\ x_{S,2}(k+1) \\ x_{T,P}(k+1) \\ x_{T,L}(k+1) \\ x_R(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{=:A} \underbrace{\begin{bmatrix} x_{S,1}(k) \\ x_{S,2}(k) \\ x_{T,P}(k) \\ x_{T,L}(k) \\ x_R(k) \end{bmatrix}}_{=:x(k)} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ f_{T,P}(x_{T,P}(k), u_{T,P}(k)) \\ 0 \\ 0 \end{bmatrix}}_{=:f_G(x(k),u(k))} +$$
$$B_{\sigma(k)} \underbrace{\begin{bmatrix} u_{T,L}(k) \\ u_S(k) \end{bmatrix}}_{=:u_B(k)} + w(k) \tag{1.13}$$

with states $x(k)$, inputs $u(k) = [u_{T,P}(k), u_{T,L}(k), u_S(k)]^\top$ and disturbance expectation

$$w^*(k) = [0,0,0,0,-1]^\top. \tag{1.14}$$

The switched input matrix $B_{\sigma(k)}$ is defined as

$$B_{\sigma(k)} \in \{B_0, B_1\}, \quad B_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ -1 & 0 \end{bmatrix},$$

together with the switching policy $\sigma(k) := x_{T,P}(k)$. The dynamics f_G of the truck are encoded in a Graph, see Fig. 1.2. The graph encodes the supply

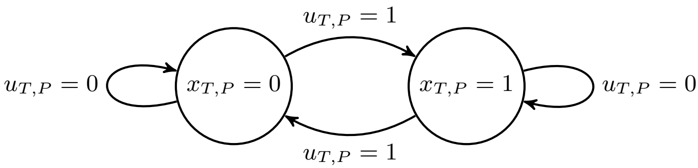


Figure 1.2: Dynamics $f_{T,P}$ of the truck's position. Possible state (position) values are represented by nodes and feasible transitions by edges.

network structure and the graph dynamics describing how the truck can travel. In short we have dynamics of the form

$$x(k+1) = Ax(k) + B_{\sigma(k)}u_B(k) + f_G(x(k),u(k)) + w(k) \tag{1.15}$$

with $x \in \mathbb{X}$ and $u \in \mathbb{U}$. We will use this representation for solving the upcoming mixed integer finite horizon control problem. Let $a \in \mathbb{R}, a > 100$ be a finite but arbitrarily large constant. The constraints are given by

$$\begin{aligned} 0 \leq x_{S,1} &\leq a \\ 0 \leq x_{S,2} &\leq a \\ 0 \leq x_{T,L} &\leq 10 \\ -a \leq x_R &\leq a. \end{aligned} \tag{1.16}$$

Note that we can also rewrite the system dynamics as

$$x(k+1) = \bar{A}x(k) + \bar{B}u(k) + w(k) \tag{1.17}$$

by incorporating the graph structure as coupled (non-convex) state and input constraints. By doing so we can e.g. state that the dynamics are continuous, which is required for various control methods. Note, that the resulting input and state constraints do not have an interior.

Stage cost

The stage cost can be divided into three parts:

1. The production and storage cost is defined by $\ell_S(x,u) := x_{S,1} + 0.5x_{S,2}$.
2. The cost for truck load and driving reads $\ell_T(x,u) := x_{T,L} + u_{T,P}$.

3. For the retail store we have a storage cost for a positive number of goods in the store. A larger demand than available goods (negative number of goods), which results in unhappiness of the customers, is modelled by a high cost. More precisely,

$$\ell_R(x,u) := \begin{cases} -10x_R, \ x_R < 0 \\ x_R, \ x_R \geq 0. \end{cases} \tag{1.18}$$

In summary we end up with the stage cost

$$\begin{aligned} \ell(x,u) &= \ell_S(x,u) + \ell_T(x,u) + \ell_R(x,u) \\ &= \begin{cases} x_{S,1} + 0.5x_{S,2} + x_{T,L} + u_{T,P} - 10x_R, \ x_R < 0 \\ x_{S,1} + 0.5x_{S,2} + x_{T,L} + u_{T,P} + x_R, \ x_R \geq 0 \end{cases} \end{aligned} \tag{1.19}$$

which is continuous and bounded on $\mathbb{X} \times \mathbb{U}$.

Identification of the optimal periodic orbit candidate

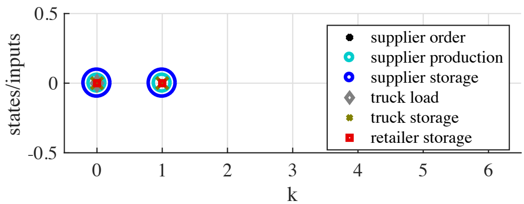
We solve for the optimal periodic orbit as described in Rem. 1.2.2 approximately in terms of finite P using the dynamics (1.13), stage cost (1.19) and the expected (nominal) disturbance (1.14). We obtain

$$\begin{aligned} &P^* = 2, \\ &\Pi^* = \left\{ \left(\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right), \left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \right) \right\} \end{aligned} \tag{1.20}$$

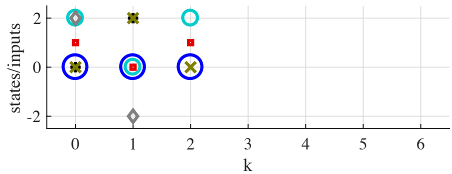
with average cost $\frac{1}{2}\sum_{k=0}^{1} \ell(x_k^{p*}, u_k^{p*}) = \tilde{\ell}_{\Pi^*} = 3.5$ along the periodic orbit. A rigorous method for verifying that (1.20) is not only an approximate but exact solution (in terms of infinitely many possible period lengths P) will be introduced in the next chapter.

Note, that for $w^*(k) = [0,0,0,0,0]^\top$ we obtain the origin ($P = 1$) as solution, see Fig. 1.3 (a). This allows us to state the educated guess, that the optimal operation depends on the disturbance in the example considered. The different resulting orbits with corresponding expected disturbance values are illustrated in Fig. 1.3.

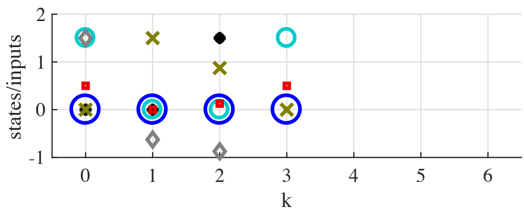
Interestingly in Fig. 1.3 (c), the truck is used as extended storage for the retail store in iterations $k = 1$ and $k = 2$. The solution in this particular case is not unique, because the storage cost for the truck and the retail store are chosen to be identically (1.19). Nevertheless, it is cheaper to use the storage (either of the truck or the retail store) than to drive more frequently between the supplier and the retail store. Fig. 1.3 (d) demonstrates that we can easily get complex periodic orbits, which also raises the question if chaotic behaviour ($P^* \to \infty$) could be optimal. This underlines the need for an analytical verification of optimal operation from both views: practical and theoretical, so that we can choose a control strategy appropriately for predictable closed loop behaviour and the best possible performance.



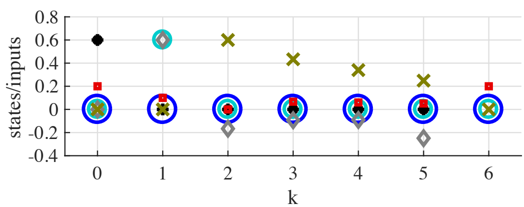
(a) Disturbance $w^* = [0{,}0{,}0{,}0{,}0]^\top$, $\bar{P} = 1$, truck position $0 \to 0.$.



(b) Disturbance $w^* = [0{,}0{,}0{,}0,-1]^\top$, $\bar{P} = 2$, truck position $0 \to 1 \to 0.$.



(c) Disturbance $w^* = [0{,}0{,}0{,}0,-0.5]^\top$, $\bar{P} = 3$, truck position $0 \to 1 \to 1 \to 0.$.



(d) Disturbance $w^* = [0{,}0{,}0{,}0,-0.1]^\top$, $\bar{P} = 6$, truck position $0 \to 0 \to 1 \to 1 \to 1 \to 1 \to 0.$.

Figure 1.3: Different approximate optimal periodic orbits for system (1.13).

2 Optimal periodic operation

2.1 Introduction

We wish to find a method in order to decide if any given candidate P-periodic orbit of a certain system class is the systems *possible best and unique* operating behaviour. By system orbit we mean a P-periodic input and state trajectory. We consider convex constrained, linear, P-periodic, time varying (LTV) systems with respect to convex cost functionals, as well as the special case of piecewise defined, convex *linear* cost functionals. We begin with a saddle point interpretation of the so called dissipativity criterion. By investigating the *uniqueness* of the saddle point, we establish the relation to strict dissipativity. For the more specialized case of piece-wise linear stage cost functionals, as in Sec. 1.3, we show that the criterion can be verified efficiently by solving two different linear programs. Last we derive the explicit structure of those linear programs.

One may ask why we need a method for verification of an optimal solution in a linear problem setting. Despite the fact that we can globally optimal solve for input and state trajectories for a fixed period length P, we can *not* easily guarantee that the solution obtained is 1. unique and 2. that there does not exist an equal or better orbit with a larger period P. Both properties are mainly interesting from an analytic point of view and we want to mention, that we can not solve the resulting online optimization problem for arbitrary long time horizons anyway. Nevertheless, recent economic model predictive control schemes e.g. [18], provide performance and asymptotic stability guarantees which depend on this property by construction. Note, that all of the results obtained can be easily applied to optimal steady state operation ($P = 1$) as well.

The approach will be introduced in several steps. Following recent literature, we introduce in Sec. 2.2 the relation between (unique) optimal periodic operation, (unique) optimal steady state operation and (strict) dissipativity with respect to a periodic orbit for general systems. In Sec. 2.3 we establish the link between strict dissipativity, strong duality and uniqueness of a saddle point in case of convex constrainted LTV systems with convex stage cost. In

Sec. 2.4 we consider the system class and stage cost from Sec. 1.3 (linear system, piece-wise linear stage cost, polytopic constraints) and provide a linear programming problem for verification of dissipativity and another linear program for verification of strict dissipativity.

2.2 Dissipativity and optimal periodic operation

The concept of dissipativity for verifying whether or not a given periodic orbit is a systems' optimal operation was first introduced in [11]. The key is the P-step system (Def. 1.2.3) that enables to leverage steady state analysis for periodic analysis. I.e. one can show that steady state operation is (under some additional controllability assumptions) optimal if and only if a certain dissipativity condition is fulfilled [3]. A consequent proof for transferring the criterion from the case of optimal steady state operation to periodic operation is given in [19]. In the following we repeat the results that are relevant for this and the following chapters.

Definition 2.2.1 (Optimal operation at Π^* [19])**.** System (1.5) is optimally operated at the feasible P^*-periodic orbit Π^* (Def. 1.2.1) if for each $x \in \mathbb{X}$ and each $u \in \mathbb{U}^\infty(x)$ the following inequality holds:

$$\liminf_{T\to\infty} \frac{\sum_{k=0}^{T-1} \ell(x_u(k,x),u(k))}{T} \geq \frac{1}{P^*} \sum_{k=0}^{P^*-1} \ell(x_k^{p*}, u_k^{p*}),$$

i.e., any feasible solution will result in an asymptotic average performance which is at most as good as the average performance of the optimal periodic orbit Π^*. Further, if system (1.5) is optimally operated at some P^* periodic orbit $\Pi^* = \{(x_0^{p*}, u_0^{p*}),\ldots,(x_{P^*-1}^{p*}, u_{P^*-1}^{p*})\}$, then Π^* is necessarily an optimal periodic orbit for system (1.5) and we have

$$\frac{1}{P^*} \sum_{k=0}^{P^*-1} \ell(x_k^{p*}, u_k^{p*}) = \inf_{P\in\mathcal{I}_{\geq 0},\Pi\in S_\Pi^P} \frac{1}{P} \sum_{k=0}^{P-1} \ell(x_k^p,u_k^p), \tag{2.1}$$

where S_Π^P denotes the set of all feasible P-periodic orbits according to Def. 1.2.1. ■

In addition there is a slightly stronger property than optimal operation at a certain optimal periodic orbit, namely suboptimal operation off periodic operation.

Definition 2.2.2 (Suboptimal operation off periodic operation [19])**.** System (1.5) is uniformly suboptimally operated off periodic operation if it is optimally operated at periodic operation and in addition there exist $\bar{\delta} > 0$ and $d \in \mathcal{K}_\infty$ such that for each $\delta > 0$ and each $\epsilon > 0$ there exists $R_{\epsilon,\delta} \in \mathcal{I}_{\geq 0}$ such that $\delta / R_{\epsilon,\delta} \geq d(\epsilon)$ for all $\delta > \bar{\delta}$ and for each $T \in \mathcal{I}_{\geq 0}$ at least one of the following conditions holds:

$$\sum_{t=0}^{TP-1} \ell(x_u(t,x),u(t)) > T \sum_{k=0}^{P-1} \ell(x_k^p,u_k^p) + \delta, \tag{2.2}$$

$$\#\{t \in \mathcal{I}_{[0,T-1]} : \sum_{j=0}^{P-1} |x_u(tP+j,x)|_{\Pi_{\mathbb{X}}} > \epsilon\} \leq R_{\epsilon,\delta}.$$

with $\#\{\mathcal{A}\}$ indicating the number of elements contained in a set $\mathcal{A}$. ■

As mentioned before, the key ingredient in this section is to establish the link between steady state and periodic operation using the P-step system from Def. 1.2.3. This is achieved by the following lemma.

Lemma 2.2.3 (Optimal operation of P-step system [19])**.** *Suppose that $\tilde{\ell}$ is bounded from below on $\mathbb{X}^P \times \mathbb{U}^P$. Then system (1.5) is optimally operated at a P^*-periodic orbit Π^* (uniformly suboptimally operated off the P^*-periodic orbit Π^*) if and only if the corresponding P^*-step system (Def. 1.2.3) is optimally operated at steady-state (uniformly suboptimally operated off steady-state).* ■

Proof. The proof can be found in [19]. □

Lemma 2.2.3 allows to use 'classical' optimal steady state analysis for periodic behaviors, as e.g. in [3], which is based on the concept of dissipativity as it was originally introduced in [7].

Definition 2.2.4 ((Strict) dissipativity with respect to a periodic orbit [11], [19])**.** The P-step system (1.9) is dissipative with respect to a periodic orbit Π and the supply rate

$$s(\tilde{x}, \tilde{u}) := \tilde{\ell}(\tilde{x}, \tilde{u}) - \sum_{k=0}^{P-1} \ell(x_k^p, u_k^p)$$

if there exists a storage function $\tilde{\lambda} : \mathbb{X}^P \to \mathbb{R}$ such that

$$\tilde{\lambda}(f^P(\tilde{x}, \tilde{u},0)) - \tilde{\lambda}(\tilde{x}) \leq s(\tilde{x}, \tilde{u}) \tag{2.3}$$

for all $(\tilde{x}, \tilde{u}) \in \tilde{\mathbb{Z}}^0$ with

$$\tilde{\mathbb{Z}}^0 := \{(\tilde{x}, \tilde{u}) \in \mathbb{X}^P \times \mathbb{U}^P | \exists \tilde{v} \in \mathbb{U}^\infty(x_{P-1}) \text{ s.t. } \tilde{v}(0) = \tilde{u}\}. \tag{2.4}$$

Furthermore it is *strictly* dissipative with respect to a periodic orbit Π if there exits a function $\alpha \in \mathcal{K}_\infty$ such that

$$\tilde{\lambda}(f^P(\tilde{x}, \tilde{u}, 0)) - \tilde{\lambda}(\tilde{x}) \leq \tilde{\ell}(\tilde{x}, \tilde{u}) - \sum_{k=0}^{P-1} \ell(x_k^p, u_k^p) - \alpha(|(\tilde{x}, \tilde{u})|_\Pi) \tag{2.5}$$

for all $(\tilde{x}, \tilde{u}) \in \tilde{\mathbb{Z}}^0$. ■

In the remainder we assume for simplicity that $\mathbb{X}^P \times \mathbb{U}^P = \tilde{\mathbb{Z}}^0$, i.e. $\mathbb{U}^\infty(\tilde{x}) \neq \emptyset \; \forall \tilde{x} \in \mathbb{X}^P$. Given the notion of dissipativity above we state the following sufficient characterization of optimal periodic orbits, inspired by steady state analysis (just for the P-step system).

Corollary 2.2.5 (Sufficient condition for optimal periodic orbits [19])**.** *Suppose that $\tilde{\ell}$ is bounded from below on $\mathbb{X}^P \times \mathbb{U}^P$ and define $\tilde{\ell}_\Pi := \sum_{k=0}^{P-1} \ell(x_k^p, u_k^p)$. Then the following statements hold.*

1. *If the P-step system from Def. (1.2.3) is dissipative on $\mathbb{X}^P \times \mathbb{U}^P$ with respect to the supply rate $s(\tilde{x}, \tilde{u}) = \tilde{\ell}(\tilde{x}, \tilde{u}) - \tilde{\ell}_\Pi$, then system (1.5) is optimally operated at the periodic orbit Π.*

2. *If the P-step system from Def. (1.2.3) is strictly dissipative on $\mathbb{X}^P \times \mathbb{U}^P$ with respect to the supply rate $s(\tilde{x}, \tilde{u}) = \tilde{\ell}(\tilde{x}, \tilde{u}) - \tilde{\ell}_\Pi$ and with a storage function $\tilde{\lambda}$ which is bounded on $\mathbb{X}^P$, then system (1.5) is uniformly suboptimally operated off the periodic orbit Π^*.*

■

Proof. The proof is due to Lem. 2.2.3 equivalent to the case in which steady state operation is optimal and can be found in [19]. □

Remark 2.2.6 (Necessary conditions)**.** *Under additional controllability assumptions, (strict) dissipativity is also a necessary condition for optimal operation at Π^* (suboptimal operation off Π^*) [19].* ■

2.3 Strong duality, uniqueness and (strict) dissipativity

Consider systems of the type

$$x(k+1) = A(k)x(k) + B(k)u(k) + w^*(k) \tag{2.6}$$
$$A_x x(k) \leq b_x, \quad (\widehat{=} x(k) \in \mathbb{X}) \quad \forall k \in \mathcal{I}_{\geq 0} \tag{2.7}$$
$$A_u u(k) \leq b_u, \quad (\widehat{=} u(k) \in \mathbb{U}) \quad \forall k \in \mathcal{I}_{\geq 0} \tag{2.8}$$

with time-varying, P-periodic matrices $A(k) = A(k+P), A(k) \in \mathbb{R}^{n\times n}$, $B(k) = B(k+P)$,$B(k) \in \mathbb{R}^{n\times m}$ and nominal periodic disturbances $w^*(k) = w^*(k+P), w^*(k) \in \mathbb{R}^n$ *without* stochasticity, with $k \in \mathcal{I}_{\geq 0}$ and $P \in \mathcal{I}_{\geq 1}$. The state (2.7) and input constraints (2.8) are assumed to be a convex polytope [5]. Note, that the state and input constraints could also be assumed to be P-periodic, time varying, which leads to more involved controllability assumptions and will therefore be neglected here for simplicity. We assume that there exists a $\mathbb{X}^0 \subseteq \mathbb{X}$ such that $\mathbb{U}(x)^\infty \neq \emptyset$ for all $x \in \mathbb{X}^0$. This is a necessary condition for existence of a feasible periodic orbit Π. System (2.6) is equipped with a continuous convex [5] stage cost function

$$\ell : \mathbb{X} \times \mathbb{U} \to \mathbb{R}. \tag{2.9}$$

Since $\mathbb{X}$ and $\mathbb{U}$ are compact and ℓ is continuous, (2.9) is bounded from below and from above. Without loss of generality we assume in addition that $0 \leq \inf_{x\in\mathbb{X},u\in\mathbb{U}} \ell(x,u)$. The stage cost function for the corresponding P-step system (1.10) is also convex, because the sum of convex functions is a convex function [5]. *Importantly*, note that the P-step system stage cost will not be necessarily strictly convex, even if the system stage cost is strictly convex. We extend the known result, that linear systems with convex stage cost functions are optimally operated at steady state, to the case of periodic operation.

Theorem 2.3.1. *System* (2.6) *with convex stage cost* (2.9) *is optimally operated at a P-periodic orbit.* ■

Proof. Consider the P-step system according to system (2.6). The P-step system stage cost function is convex. The dynamics of the P-step system is time invariant, because the dynamics of system (2.6) is P-periodic. Any P-periodic orbit of system (2.6) is a steady state of the P-step system. Therefore by [2, Thm. 4] the P-step system is optimally operated at steady state. By Lem. 2.2.3 it follows that system (2.6) is optimally operated at a corresponding P-periodic orbit. □

Remark 2.3.2. *The result given above is particularly useful in the sense, that it is sufficient to solve* (2.1) *for period length P only. By Thm. 2.3.1 the solution must be the (possible non-unique) best operating behavior of system* (2.6). ■

The fact that the P-step system stage cost function is not necessarily strictly convex is unfortunate, because in case of strictly convex stage cost functionals we can *not* conclude suboptimal operation off periodic operation, analogue to the case of steady-state operation as e.g. in [2] or [8]. In the remainder of this chapter we aim at building a framework for checking suboptimal operation off a given periodic orbit via strict dissipativity. Given a P periodic orbit Π, consider

$$(P_{\text{orbit}}) \begin{cases} \min_{\tilde{x}\in\mathbb{X}^P,\tilde{u}\in\mathbb{U}^P} \tilde{\ell}(\tilde{x},\tilde{u}) - \tilde{\ell}_\Pi \\ \text{s.t.} \quad \tilde{x} = f^P(\tilde{x},\tilde{u},\tilde{w}^*) \end{cases}$$

with f^P the P-step system dynamics of (2.6), $\tilde{\ell}_\Pi$ the cost along Π, and $\tilde{w}^* = (w_0^*,..,w_{P-1}^*) = (w^*(0),..,w^*(P-1))$. The optimal solution is denoted by $\tilde{x}^*$, $\tilde{u}^*$. The corresponding Lagrangian reads

$$L(\tilde{x},\tilde{u},\tilde{\nu}) = \tilde{\nu}^T(\tilde{x} - f^P(\tilde{x},\tilde{u},\tilde{w}^*)) + \tilde{\ell}(\tilde{x},\tilde{u}) - \tilde{\ell}_\Pi \tag{2.10}$$

with Lagrange multipliers $\tilde{\nu} \in \mathbb{R}^{nP}$ and Lagrange multipliers $\tilde{\nu}^* \in \mathbb{R}^{nP}$ at the saddle point, see e.g. [5] for a definition.

Theorem 2.3.3 (Strong duality and dissipativity)**.** *The following statements are equivalent:*

(1) The LTV system (2.6) *with stage cost* (2.9) *is dissipative with respect to a periodic orbit Π^* and a linear storage function $\tilde{\lambda}^\top\tilde{x}$ according to* (2.3).

(2) $$\begin{aligned} 0 &\le L(\tilde{x}^*,\tilde{u}^*,\tilde{\nu}^*) \\ &\le \min_{\tilde{x}\in\mathbb{X}^P,\tilde{u}\in\mathbb{U}^P} \left(\max_{\tilde{\nu}} L(\tilde{x},\tilde{u},\tilde{\nu})\right) \\ &\le \max_{\tilde{\nu}} \left(\min_{\tilde{x}\in\mathbb{X}^P,\tilde{u}\in\mathbb{U}^P} L(\tilde{x},\tilde{u},\tilde{\nu})\right). \end{aligned}$$

■

Proof. (1) $\Rightarrow$ (2): Let $\tilde{x}^*$,$\tilde{u}^*$ describe a state and input trajectory of system (2.6) according to Π^*. We have for all $\tilde{\nu} \in \mathbb{R}^{nP}$ that $L(\tilde{x}^*,\tilde{u}^*,\tilde{\nu}) = 0$ by stationarity w.r.t. the P-step system. By inserting $\tilde{\lambda}$ from (1) we have $0 \le$

$L(\tilde{x}, \tilde{u}, \tilde{\lambda})$ for all $\tilde{x} \in \mathbb{X}^P, \tilde{u} \in \mathbb{U}^P$ by the dissipativity equation and conclude that minimizing $L(\tilde{x}, \tilde{u}, \tilde{\lambda})$ w.r.t. feasible $\tilde{x}, \tilde{u}$ yields exactly 0. By choosing $\tilde{\nu}^* = \tilde{\lambda}$ we have shown that $0 \leq L(\tilde{x}^*, \tilde{u}^*, \tilde{\nu}^*)$ holds[1]. Further, since the weak slater condition [5, chapter 5.2] (affine constraints, convex cost) holds, we have strong duality and therefore it follows that the dual problem has the same solution as the primal problem [5] and consequently both optimization problems in (2) have equal solutions.

(2) $\Rightarrow$ (1): Because of strong duality there exists a constant maximizer $\tilde{\nu}^*$ such that

$$
\begin{aligned}
&0 \leq \min_{\tilde{x} \in \mathbb{X}^P, \tilde{u} \in \mathbb{U}^P} L(\tilde{x}, \tilde{u}, \tilde{\nu}^*) \\
&\Rightarrow 0 \leq L(\tilde{x}, \tilde{u}, \tilde{\nu}^*), \forall \tilde{x} \in \mathbb{X}^P, \tilde{u} \in \mathbb{U}^P.
\end{aligned}
$$

Let $\tilde{\lambda} = \tilde{\nu}$ which fulfill (1) and therefore the proof is complete. □

Remark 2.3.4. *Thm. 2.3.3 recovers the same result as [2, Thm. 4], namely that any linear system with respect to convex cost functionals is optimally operated at steady state as described also in Thm. 2.3.1.* ■

Theorem 2.3.5 (Strict dissipativity based on uniqueness)**.** *Let system* (2.6) *be dissipative with respect to the supply rate $\tilde{\ell}(\tilde{x},\tilde{u}) - \tilde{\ell}_{\Pi^*}$ according to stage cost* (2.9) *and the P-periodic orbit Π^*. Then the following statements are equivalent.*

(1) System (2.6) *with stage cost* (2.9) *is strictly dissipative with respect to Π^*.*

(2) All minimizers of (P_{orbit}) are elements of $\tilde{\Pi}^*$. ■

Proof. (1) $\Rightarrow$ (2): We show the implication by contradiction, i.e. let (1) hold and $\tilde{\Pi}^*$ contains *not* all elements of the set of minimizers of (P_{orbit}). We conclude that there must exists a $(\bar{x}, \bar{u}) \notin \tilde{\Pi}^*$ that also minimizes (P_{orbit}). From the equality (stationarity) constraint in (P_{orbit}) it follows that that $\bar{x} = f^P(\bar{x}, \bar{u}, 0)$ holds. Inserting $(\bar{x}, \bar{u}, 0)$ in the general strict dissipativity condition (2.5) yields

$$
0 \leq -\alpha_\ell(|(\bar{x}, \bar{u})|_{\Pi^*}).
$$

[1] We will need the inequality later for the case of piece-wise defined cost functions (instead of just equality to zero). In this case we analyze dissipativity only on a subset and therefore it could happen that the optimal orbit is not included on that subset.

The statement above is false, because if $(\bar{x}, \bar{u}) \notin \tilde{\Pi}^*$ then the distance $|(\bar{x}, \bar{u})|_{\Pi^*}$ must be greater than zero and together with $\alpha \in \mathcal{K}_\infty$, the dissipativity inequality does not hold, which is the desired contradiction.

(2) $\Rightarrow$ (1): Let $\tilde{x}^*, \tilde{u}^* \in \tilde{\Pi}^*$. (P_{orbit}) attains zero for $\tilde{x}^*, \tilde{u}^*$, because $\tilde{x}^*, \tilde{u}^*$ corresponds to Π^*. We have by Thm. 2.3.3 that there exists a $\tilde{\nu}^* \in \mathbb{R}^{nP}$ for dissipativity such that

$$0 \leq L(\tilde{x}, \tilde{u}, \tilde{\nu}^*) \text{ for all } \tilde{x} \in \mathbb{X}^P, \tilde{u} \in \mathbb{U}^P.$$

By statement (2) and strong duality (weak slater condition) it follows for all $(\tilde{x}, \tilde{u}) \in \mathbb{X}^P \times \mathbb{U}^P$ and $(\tilde{x}, \tilde{u}) \notin \tilde{\Pi}^*$ (not a minimizer of (P_{orbit}))

$$0 < L(\tilde{x}, \tilde{u}, \tilde{\nu}^*).$$

Therefore, L is positive definite in $\tilde{x}, \tilde{u}$ with respect to $\tilde{\Pi}^*$. Define as in [13, p. 341] the non-decreasing function

$$\hat{\alpha}(r) := \min_{\tilde{x}, \tilde{u} \in \mathbb{X}^P \times \mathbb{U}^P : |\tilde{x}, \tilde{u}|_{\Pi^*} = r} L(\tilde{x}, \tilde{u}, \tilde{\nu}^*)$$

which is continuous at $r = 0$ and has the properties $\hat{\alpha}(0) = 0$, $\hat{\alpha}(r) > 0$ for $r > 0$, and

$$\hat{\alpha}(|\tilde{x}, \tilde{u}|_{\Pi^*}) \leq L(\tilde{x}, \tilde{u}, \tilde{\nu}^*) \text{ for all } \tilde{x} \in \mathbb{X}^P, \tilde{u} \in \mathbb{U}^P.$$

Analogously to [13, p.341] these properties imply that there exists a function $\alpha_1 \in \mathcal{K}$ with the same properties as $\hat{\alpha}$ which is *strictly* increasing. Consider the $(n+m)P$ dimensional Ball $B_{\mathbb{X}^P \times \mathbb{U}^P}$ with a chebyshev center [10] and chebyshev radius $r_{\mathbb{X}^P \times \mathbb{U}^P}$ such that $\mathbb{X}^P \times \mathbb{U}^P \subset B_{\mathbb{X}^P \times \mathbb{U}^P}$. Define

$$\alpha(r) = \begin{cases} \alpha_1(r), \; r \leq 2r_{\mathbb{X}^P \times \mathbb{U}^P} \\ r - 2r_{\mathbb{X}^P \times \mathbb{U}^P} + \alpha_1(2r_{\mathbb{X}^P \times \mathbb{U}^P}), \, r > 2r_{\mathbb{X}^P \times \mathbb{U}^P} \end{cases}$$

which is strictly increasing, $\lim_{r \to \infty} \alpha(r) = \infty$, continuous and therefore element of $\mathcal{K}_\infty$. Further by the definition of α_1 it holds

$$\alpha(|\tilde{x}, \tilde{u}|_{\Pi^*}) \leq L(\tilde{x}, \tilde{u}, \tilde{\nu}^*) \text{ for all } \tilde{x} \in \mathbb{X}^P, \tilde{u} \in \mathbb{U}^P$$

which equals the strict dissipativity inequality equation with respect to Π^* and storage function $\tilde{\nu}^{*\top} \tilde{x}$. We have shown the equivalence of the two statements in Thm. 2.3.5, hence the proof is complete. $\square$

Piecewise defined cost functionals and (strict) dissipativity

Checking dissipativity can be done by a 'divide and conquer' strategy in case of piece-wise defined convex cost functionals. This enables us to keep the size of the resulting optimization problems small while having a linearly increasing number of such optimization problems for each region. Thm. 2.3.3 for verifying dissipativity can also be applied in case of a convex piecewise defined, continuous stage cost $\mathbb{R}^n \to \mathbb{R}$, defined as

$$\ell(x,u) = \begin{cases} \ell_1(x,u), & \forall x, u \in \mathbb{L}_1, \\ \ell_2(x,u), & \forall x, u \in \mathbb{L}_2, \\ \vdots & \\ \ell_{n_L}(x,u), & \forall x, u \in \mathbb{L}_{n_L} \end{cases} \tag{2.11}$$

with disjoint sets $\mathbb{L}_1$, $\mathbb{L}_2$, ..., $\mathbb{L}_{n_L} \subset \mathbb{L}$, $\bigcup_{i=1}^{n_L} \mathbb{L}_i = \mathbb{L} \supseteq \mathbb{X} \times \mathbb{U}$, i.e. $\mathbb{L}_i \cap \mathbb{L}_j = \emptyset \; \forall i \neq j,\; i,j \in \mathcal{I}_{[1,n_L]}$. The corresponding stage cost of the P-step system (1.10) is then defined with $n_{\tilde{L}} = n_L^P$ different convex functions[2]. For easier reference of the individual cases of $\tilde{\ell}$ we introduce the notation

$$\tilde{\ell}(\tilde{x}, \tilde{u}) = \begin{cases} \tilde{\ell}_1(\tilde{x},\tilde{u}), & \forall \tilde{x}, \tilde{u} \in \tilde{\mathbb{L}}_1, \\ \tilde{\ell}_2(\tilde{x}, \tilde{u}), & \forall \tilde{x}, \tilde{u} \in \tilde{\mathbb{L}}_2, \\ \vdots & \\ \tilde{\ell}_{n_{\tilde{L}}}(\tilde{x}, \tilde{u}), & \forall \tilde{x}, \tilde{u} \in \tilde{\mathbb{L}}_{n_{\tilde{L}}} \end{cases} \tag{2.12}$$

where $\tilde{\mathbb{L}}_1$, $\tilde{\mathbb{L}}_2$, ..., $\tilde{\mathbb{L}}_{n_{\tilde{L}}} \subset \tilde{\mathbb{L}}$, $\bigcup_{i=1}^{n_{\tilde{L}}} \tilde{\mathbb{L}}_i = \tilde{\mathbb{L}} \supseteq \mathbb{X}^P \times \mathbb{U}^P$ and $\tilde{\mathbb{L}}_i \cap \tilde{\mathbb{L}}_j = \emptyset \; \forall i \neq j,\; i,j \in \mathcal{I}_{[1,n_{\tilde{L}}]}$ on which the P-step system stage cost function (2.12) is defined. In the next section, we give an explicit representation of $\tilde{\mathbb{L}}_i$ in case of a piecewise linear stage cost function. Note that if (2.11) is convex, (2.12) will also be convex, because the sum of convex functions is a convex function [5].

Corollary 2.3.6 (Piecewise treatment for dissipativity). *The following statements are equivalent:*

(1) System (2.6) *with stage cost* (2.11) *is dissipative with respect to the supply rate* $\tilde{\ell}(\tilde{x},\tilde{u}) - \tilde{\ell}_\Pi$ *according to* Π *and storage function* $\tilde{\lambda}^\top \tilde{x}$ *on* $\tilde{\mathbb{L}}$*.*

[2] For every state-input pair $(\tilde{x}, \tilde{u})$ of the P-step system we have a sum over P stage costs, see (1.10). Consequently, for each term of the sum there are n_L cases and because of P terms we have n_L^P cases in total.

(2) *System* (2.6) *with stage cost* (2.11) *is dissipative with respect to the supply rate* $\tilde{\ell}(\tilde{x},\tilde{u}) - \tilde{\ell}_{\Pi}$ *according to* Π *and storage function* $\tilde{\lambda}_i^\top \tilde{x}$ *on each* $\tilde{\mathbb{L}}_i$*, for all* $i \in \mathcal{I}_{[1,n_{\tilde{L}}]}$.

■

Proof. (1) ⇒ (2): Choose $\tilde{\lambda}_i := \tilde{\lambda}$ from (1).

(2) ⇒ (1): For a contradiction we investigate (2) $\wedge \neg$ (1). By Thm. 2.3.3 we know that if $\neg$ (1) then there must exist constant $\tilde{x} \in \mathbb{X}$, $\tilde{u} \in \mathbb{U}$ such that $0 > \max_{\tilde{\nu}} L(\tilde{x}, \tilde{u}, \tilde{\nu})$. Choosing $\tilde{\nu}^* = \tilde{\lambda}_i$ with λ_i corresponding to the region $(\tilde{x}, \tilde{u}) \in \tilde{\mathbb{L}}_i$ yields $0 \leq L(\tilde{x}, \tilde{u}, \tilde{\nu}^*)$. Thus we have the desired contradiction. □

Corollary 2.3.7 (Piecewise treatment for strict dissipativity)**.** *Cor. 2.3.6 holds in case of investigating strict dissipativity with respect to the supply rate* $\tilde{\ell}(\tilde{x},\tilde{u}) - \tilde{\ell}_{\Pi}$ *according to the stage cost* (2.11) *and* Π *as well.* ■

Proof. The first part of the proof can be exactly constructed as the proof of Cor. 2.3.6 by additionally considering the existence of a function $\alpha \in \mathcal{K}_\infty$. For the second part note, that if each region is strictly dissipative, by Thm. 2.3.5 the solution of (P_{orbit}) for each region must lie in $\tilde{\Pi}$. This implies that the minimizers of (P_{orbit}) lie in $\tilde{\Pi}$ and therefore, again by Thm. 2.3.5, we have strict dissipativity on $\tilde{\mathbb{L}}$. □

2.4 Linear cost functionals and (strict) dissipativity

Let system (2.6) be equipped with a linear stage cost functional $\ell : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$ defined as

$$\ell(x,u) = \ell_x^\top x + \ell_u^\top u \tag{2.13}$$

with $\ell_x \in \mathbb{R}^n$ and $\ell_u \in \mathbb{R}^m$. Note, that since $\mathbb{X}$ and $\mathbb{U}$ are compact and ℓ is continuous, (2.13) is bounded from below and from above[3]. The stage cost function (1.10) for the corrsponding P-step system reads

$$\tilde{\ell}(\tilde{x}, \tilde{u}) := \sum_{j=0}^{P-1} (\ell_x^\top x_u(j, x_{P-1}) + \ell_u^\top u_j). \tag{2.14}$$

[3]Without loss of generality we could assume in addition that $0 \leq \inf_{x \in \mathbb{X}, u \in \mathbb{U}} l(x, u)$.

An explicit dissipativity inequality

In order to obtain a compact explicit representation of the dissipativity inequality (2.3) in case of system (2.6) with stage cost (2.13) we establish the following auxiliary result for general systems (1.5) with general stage cost (1.6).

Lemma 2.4.1 (P-step system (strict) dissipativity on dynamics manifold). *Consider the manifold*

$$\mathcal{M} := \{(\tilde{x},\tilde{u}) \in \mathbb{X}^P \times \mathbb{U}^P | x_{k+1} = f(x_k,u_k,0),\, k \in \mathcal{I}_{[0,P-2]}\}. \tag{2.15}$$

System (2.6) *is (strictly) dissipative with respect to the supply rate* $\tilde{\ell}(\tilde{x},\tilde{u}) - \tilde{\ell}_\Pi$ *on* $\mathcal{M}$. $\Leftrightarrow$ *Cor. 2.2.5 holds.* ■

Proof. '$\Leftarrow$': (Strict) dissipativity of the P-step system for all $(\tilde{x},\tilde{u}) \in \mathbb{X}^P \times \mathbb{U}^P$ implies that (strict) dissipativity also holds on the *strict* subset $\mathcal{M} \subset \mathbb{X}^P \times \mathbb{U}^P$.

'$\Rightarrow$': Note that the state space of the P-step system in Def. 1.2.3 is intrinsically restricted to the manifold $\mathcal{M}$. I.e. let $(\tilde{x}^*,\tilde{u}^*) \in \mathbb{X}^P \times \mathbb{U}^P$, assume $\exists j \in \mathcal{I}_{[0,P-2]}$ such that for the j-th element of $(\tilde{x}^*,\tilde{u}^*)$ it holds that $x_{j+1} \neq f(x_j,u_{j+1},0)$. This yields to a contradiction to the definition of the P-step system (1.8). We conclude that all implications of [19, Lem. 13] still hold, because the poof is based on a limit inspection of the P-step system in which it is intrinsically ensured that we stay on $\mathcal{M}$ as shown above. In particular Cor. 2.2.5 in [19] still holds if we restrict the (strict) dissipativity condition to $\mathcal{M}$. □

Remark 2.4.2. *Lem. 2.4.1 loosens the original (strict) dissipativity conditions given e.g. in [18], [19], because we have to verify* (2.3), (2.5) *only on* $\mathcal{M}$ *which is a strict subset of* $\mathbb{X}^P \times \mathbb{U}^P$. *In the following, especially in Ch. 3 and Ch. 4 we use the strict dissipativity property only in cases in which the* P*-step system state is intrinsically restricted to be element of* (2.15). ■

For an explicit representation of (2.3), we build a P-step system state $\tilde{x}$ using the system dynamics (2.6) and the predecessor inputs $\tilde{u}^-$ in order to restrict the dissipativity inequality to (2.15)[4], compare also with Fig. 2.1. The

[4] The benefit is significant smaller number of variables in the the resulting linear programming problem for verifying dissipativity.

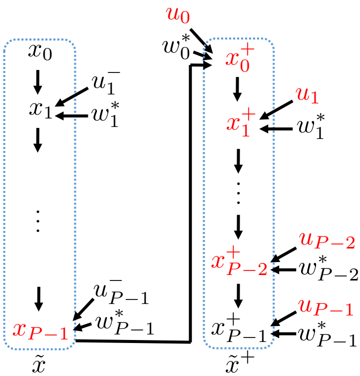


Figure 2.1: Expressing the (strict) dissipativity equation (2.5) on the the manifold (2.15). All the relevant parts of the P-step system stage cost function are red.

states can be written as

$$\tilde{x} = \begin{bmatrix} x_0 \\ A(0)x_0 + B(0)u_1^- + w_1^* \\ A(1)A(0)x_0 + A(1)B(0)u_1^- + A(1)w_1^* + B(1)u_2^- + w_2^* \\ \vdots \\ \dots \end{bmatrix} \tag{2.16}$$

in terms of inputs and disturbances. From (2.16) we infer a general formula for the j-th state of a P-step system state, with $j \in \mathcal{I}_{[1,P-1]}$:

$$x_j = \underbrace{\left[\prod_{i_2=0}^{j-1} A(j-1-i_2)\right]}_{=:\mathcal{A}_{j-1}^{0}} x_0 + \sum_{i=0}^{j-2}\left[\underbrace{\left(\prod_{i_2=0}^{j-i-2} A(j-1-i_2)\right)}_{=:\mathcal{A}_{j-1}^{i+1}} \left[B(i)u_{i+1}^{-} + w_{i+1}^{*}\right]\right] + B(j-1)u_j^{-} + w_j^{*}$$

and more compactly as

$$x_j = \mathcal{A}_{j-1}^{0} x_0 + \sum_{i=0}^{j-2}[\mathcal{A}_{j-1}^{i+1}\left(B_i u_{i+1}^{-} + w_{i+1}^{*}\right)] + B_{j-1}u_j^{-} + w_j^{*}. \tag{2.17}$$

In (2.17) we introduced the operator $\mathcal{A}_{N_1}^{N_2} : \mathbb{R}^2 \to \mathbb{R}^{n\times n}$ defined as $\mathcal{A}_{N_1}^{N_2} = \prod_{i_2=0}^{N_1-N_2} A(N_1 - i_2)$ and $B_i = B(i)$. By rewriting (2.17) in vector form with $\mathbb{I}_n$ the identity matrix and $\mathbb{O}_n$ the zero matrix of dimension $n \times n$ as

$$\tilde{x} = \underbrace{\begin{bmatrix} \mathbb{I}_n \\ \mathcal{A}_0^0 \\ \mathcal{A}_1^0 \\ \vdots \\ \mathcal{A}_{P-2}^0 \end{bmatrix}}_{=:\Omega} x_0 + \begin{bmatrix} \mathbb{O}_n \\ \mathbb{I}_n \\ \mathcal{A}_1^1 \\ \vdots \\ \mathcal{A}_{P-2}^1 \end{bmatrix} (B_0 u_1^{-} + w_1^{*}) + .. + \begin{bmatrix} \mathbb{O}_n \\ \mathbb{O}_n \\ \mathbb{O}_n \\ \vdots \\ \mathbb{I}_n \end{bmatrix} (B_{P-2}u_{P-1}^{-} + w_{P-1}^{*})$$

we define the compact expression:

$$
\begin{aligned}
\tilde{x} =& \Omega x_0 \\
&+ \underbrace{\begin{bmatrix} \mathbb{O}_n & \mathbb{O}_n & \mathbb{O}_n & \dots & \mathbb{O}_n \\ \mathbb{I}_n & \mathbb{O}_n & \mathbb{O}_n & \dots & \mathbb{O}_n \\ \mathcal{A}_1^1 & \mathbb{I}_n & \mathbb{O}_n & \dots & \mathbb{O}_n \\ \mathcal{A}_2^1 & \mathcal{A}_2^2 & \mathbb{I}_n & \dots & \mathbb{O}_n \\ & & \vdots & & \\ \mathcal{A}_{P-3}^1 & \mathcal{A}_{P-3}^2 & \mathcal{A}_{P-3}^3 & \dots & \mathbb{I}_n \end{bmatrix}}_{=:\Gamma} \begin{bmatrix} (B_0 u_1^- + w_1^*) \\ (B_1 u_2^- + w_2^*) \\ (B_2 u_3^- + w_3^*) \\ \vdots \\ (B_{P-2} u_{P-1}^- + w_{P-1}^*) \end{bmatrix} \\
=& \Omega x_0 + \Gamma \underbrace{\begin{bmatrix} B_0 & \mathbb{O}_n & \mathbb{O}_n & \dots \\ \mathbb{O}_n & B_1 & \mathbb{O}_n & \dots \\ \mathbb{O}_n & \mathbb{O}_n & B_2 & \dots \\ \mathbb{O}_n & \mathbb{O}_n & \mathbb{O}_n & \dots \\ & \vdots & \ddots & \\ \mathbb{O}_n & \mathbb{O}_n & \dots & B_{P-2} \end{bmatrix}}_{=:\mathcal{B}} \underbrace{\begin{bmatrix} u_1^- \\ u_2^- \\ u_3^- \\ \vdots \\ u_{P-1}^- \end{bmatrix}}_{=:U^-} + \Gamma \underbrace{\begin{bmatrix} w_1^* \\ w_2^* \\ w_3^* \\ \vdots \\ w_{P-1}^* \end{bmatrix}}_{=:W} \\
=& \Omega x_0 + \Gamma \mathcal{B} U^- + \Gamma W.
\end{aligned} \tag{2.18}
$$

As required, (2.18) expresses one P-step system state on the manifold $\mathcal{M}$. In order to construct an explicit representation corresponding to (2.3) we analogously derive an explicit representation of $\tilde{x}^+$. While the first element of $\tilde{x}$ is assumed to be given as mentioned above, the first element of $\tilde{x}^+$ reads

$$x_0^+ = A(P-1)x_{P-1} + B(P-1)u_0 + w_0^*, \tag{2.19}$$

as illustrated in Fig. 2.1. All the variables for an explicit representation of (2.3) on $\mathcal{M}$ can be summarized in the vector

$$Y := [x_0^\top, U^{-\top}, u_0^\top, U^\top]^\top.$$

We write (2.18) as

$$\tilde{x} = \underbrace{[\Omega, \Gamma\mathcal{B}, 0, 0]}_{=:G} Y + \Gamma W = GY + \Gamma W. \tag{2.20}$$

For a similar representation of $\tilde{x}^+$ using Y, the first element of $\tilde{x}^+$ (compare with (2.19)) is given by

$$x_0^+ = A(P-1)[GY + \Gamma W]_{P-1} + B(P-1)u_0 + w_0^*$$

where $[\tilde{x}]_{P-1}$ are the last n rows of $\tilde{x}$. Now it is straightforward to state analogue to (2.20)

$$\begin{aligned}
\tilde{x}^+ =&\Omega x_0^+ + \Gamma\mathcal{B}U + \Gamma W\\
=&\Omega A(P-1)[GY]_{P-1} + \Omega B(P-1)u_0 + \Gamma\mathcal{B}U\\
&+ \Omega A(P-1)[\Gamma W]_{P-1} + \Omega w_0^* + \Gamma W\\
=&\underbrace{\left(\Omega A(P-1)[G]_{P-1} + [0,0,\Omega B(P-1),\Gamma\mathcal{B}]\right)}_{=:G^+} Y\\
&+ \Omega A(P-1)[\Gamma W]_{P-1} + \Omega w_0^* + \Gamma W\\
=&G^+ Y + \Omega A(P-1)[\Gamma W]_{P-1} + \Omega w_0^* + \Gamma W.
\end{aligned}$$

For the constraints and stage cost in (2.3) we also need the inputs that read

$$\begin{aligned}
U^- &= \underbrace{\begin{bmatrix}\mathbb{O}_{m(P-1)\times n} & \mathbb{I}_{m(P-1)} & \mathbb{O}_{m(P-1)\times m} & \mathbb{O}_{m(P-1)}\end{bmatrix}}_{E^-} Y\\
&= E^- Y
\end{aligned}$$

with $\mathbb{O}_{n\times m}$ an n by m zeros matrix and

$$\begin{aligned}
\tilde{u} &= \underbrace{\begin{bmatrix}\mathbb{O}_{m\times n} & \mathbb{O}_{m\times m(P-1)} & \mathbb{I}_m & \mathbb{O}_{m\times m(P-1)}\\ \mathbb{O}_{m(P-1)\times n} & \mathbb{O}_{m(P-1)} & \mathbb{O}_{m(P-1)\times m} & \mathbb{I}_{m(P-1)}\end{bmatrix}}_{E} Y\\
&= EY
\end{aligned} \tag{2.21}$$

respectively. Given the equations above, the dissipativity inequality can be explicitly stated as

$$0 \leq q_0^{\tilde{\lambda}}(Y) \tag{2.22}$$

with

$$\begin{aligned}
q_0^{\tilde{\lambda}}(Y) :=&\tilde{\lambda}(GY + \Gamma W) - \tilde{\lambda}(G^+ Y + \Omega A(P-1)[\Gamma W]_{P-1}\\
&+ \Omega w_0^* + \Gamma W) + \tilde{\ell}_x^\top G_{\tilde{x}_{\tilde{\ell}_x}} Y + \tilde{\ell}_u^\top EY - \tilde{\ell}_\Pi
\end{aligned}$$

and[5] $G_{\tilde{x}_{\tilde{\ell}_x}} := \left[[G]_{P-1}^\top, [G^+]_{0:P-2}^\top\right]^\top$. In order to define the constraint set $\mathbb{X}^P \times \mathbb{U}^P$ (2.7), (2.8) in terms of Y, on which (2.22) must hold, we write the

[5] The expression $[\tilde{x}^+]_{0:P-2}$ represents the first $(P-1)n$ rows of $\tilde{x}^+$.

constraints of $\tilde{x}$ as

$$\begin{aligned}(\mathbb{I}_P \otimes A_x)\tilde{x} \leq \mathbf{1}_P \otimes b_x \Leftrightarrow \underbrace{(\mathbb{I}_P \otimes A_x)G}_{=:A_{\tilde{x}}} Y \leq \underbrace{\mathbf{1}_P \otimes b_x - (\mathbb{I}_P \otimes A_x)\Gamma W}_{=:b_{\tilde{x}}} \\ \Leftrightarrow A_{\tilde{x}} Y \leq b_{\tilde{x}}\end{aligned}$$

with $\mathbf{1}_n$ a column vector with n elements equal to one. For $\tilde{x}^+$ we define

$$\begin{aligned}A_{\tilde{x}+} &:= (\mathbb{I}_P \otimes A_x)G^+, \\ b_{\tilde{x}+} &:= \mathbf{1}_P \otimes b_x - (\mathbb{I}_P \otimes A_x)(\Omega A(P-1)[\Gamma W]_{P-1} + \Omega w_0^* + \Gamma W)\end{aligned}$$

and end up with

$$A_{\tilde{x}+} Y \leq b_{\tilde{x}+}.$$

Analogously for the inputs we have for U^-

$$\begin{aligned}(\mathbb{I}_{P-1} \otimes A_u)U^- \leq \mathbf{1}_{P-1} \otimes b_u \Leftrightarrow \underbrace{(\mathbb{I}_{P-1} \otimes A_u)E^-}_{=:A_{\tilde{u}}} Y \leq \underbrace{\mathbf{1}_{P-1} \otimes b_u}_{=:b_{\tilde{u}}} \\ \Leftrightarrow A_{\tilde{u}} Y \leq b_{\tilde{u}}\end{aligned}$$

and for $\tilde{u}$

$$\begin{aligned}&\underbrace{(\mathbb{I}_P \otimes A_u)E}_{=:A_{\tilde{u}+}} Y \leq \underbrace{\mathbf{1}_P \otimes b_u}_{=:b_{\tilde{u}+}} \\ \Leftrightarrow &A_{\tilde{u}+} Y \leq b_{\tilde{u}+}.\end{aligned}$$

Altogether we get the following constraints

$$\begin{aligned}\underbrace{\begin{bmatrix} A_{\tilde{x}} \\ A_{\tilde{x}+} \\ A_{\tilde{u}} \\ A_{\tilde{u}+} \end{bmatrix}}_{A_Y} Y \leq \underbrace{\begin{bmatrix} b_{\tilde{x}} \\ b_{\tilde{x}+} \\ b_{\tilde{u}} \\ b_{\tilde{u}+} \end{bmatrix}}_{b_Y} \\ \Leftrightarrow A_Y Y \leq b_Y.\end{aligned} \tag{2.23}$$

For easier reference we define

$$\mathbb{Y} := \{Y | A_Y Y \leq b_Y\}. \tag{2.24}$$

In summary, verifying (2.3) for system (2.6) with stage cost (2.13) is equal to verifying

$$q_0^{\tilde{\lambda}}(Y) \geq 0 \text{ for all } Y \in \mathbb{Y} \tag{2.25}$$

for a given storage function $\tilde{\lambda}$.

A linear programming formulation for (strict) dissipativity

Based on the results of Sec. 2.3 we establish the link between dissipativity and a certain linear programming (LP) problem for the system setup in this Section. Consider

$$(P_{\text{orbit, lin}}) = \begin{cases} \min_Y & \left(\tilde{\ell}_x^\top G_{\tilde{x}_{\tilde{\ell}_x}} + \tilde{\ell}_u^\top E\right) Y \\ \text{s.t.} & (G - G^+)Y = \Omega A(P-1)[\Gamma W]_{P-1} + \Omega w_0^* \\ & A_Y Y \leq b_Y \end{cases}$$

with A_Y and b_Y from (2.23).

Proposition 2.4.3 (Linear program and dissipativity)**.** *From Thm. 2.3.3 it directly follows that we can verify dissipativity of system (2.6) with respect to the supply rate $\tilde{\ell}(\tilde{x},\tilde{u}) - \tilde{\ell}_{\Pi^*}$ according to the stage cost (2.13) and a periodic orbit Π^* by solving $(P_{\text{orbit, lin}})$ and check if the solution is greater or equal to $\tilde{\ell}_\Pi$ ($\Rightarrow$ dissipative) or not ($\Rightarrow$ not dissipative).* ■

Consider the general linear programming problem

$$\begin{aligned} \min_Y \quad & p^\top Y \\ \text{s.t.} \quad & AY = b \\ & CY \geq d \end{aligned} \tag{2.26}$$

and its dual problem

$$\begin{aligned} \max_v \quad & b^\top u + d^\top v \\ \text{s.t.} \quad & A^\top u + C^\top v = p \\ & v \geq 0. \end{aligned} \tag{2.27}$$

Further let $v^* \in \mathbb{R}^{nP}$ be a solution of (2.27) and define

$$\begin{aligned} K &:= \{i | v_i^* > 0\} \\ L &:= \{i | C_i Y = d_i, v_i^* = 0\} \end{aligned}$$

where v_i^* represents the i-th row of v^*.

Corollary 2.4.4. *(Condition for strict dissipativity) Consider*

$$p^\top := \left(\tilde{\ell}_x^\top G_{\tilde{x}_{\tilde{\ell}_x}} + \tilde{\ell}_u^\top E\right),$$

$A := (G - G^+)$*,* $b := \Omega A(P-1)[\Gamma W]_{P-1} + \Omega w_0^*$*,* $C := -A_Y$*, and* $d := -b_Y$*. If and only if the rows of* $[A^\top \; C_K^\top \; C_L^\top]$ *are linearly independent, and the linear program*

$$\begin{aligned} \max_{Y} \quad & \mathbf{1}^\top C_L Y \\ \text{s.t.} \quad & AY = 0 \\ & C_K Y = 0 \\ & C_L Y \geq 0 \end{aligned} \tag{2.28}$$

has a zero maximum, then system (2.6) *with stage cost* (2.13) *is strictly dissipative with respect to the supply rate* $\tilde{\ell}(\tilde{x},\tilde{u}) - \tilde{\ell}_\Pi$ *obtained by solving* $(P_{\text{orbit,lin}})$*.* ■

Proof. System (2.6) with stage cost (2.13) is dissipative by Thm. 2.3.1. Due to the P-periodic time varying system dynamics (2.6) and the explicit, time invariant P-step system representation in $(P_{\text{orbit, Lin}})$, a shifted optimal trajectory will not be the minimizer of $(P_{\text{orbit, Lin}})$. Therefore, if the minimizer of $(P_{\text{orbit, Lin}})$ is element of $\tilde{\Pi}^*$, it remains by Thm. 2.3.5 to verify, if it is the unique solution to $(P_{\text{orbit, Lin}})$. From [15, Theorem 2, (v) and Remark 2] we know, that if and only if (2.28) has a zero maximum, the linear program $(P_{\text{orbit, lin}})$ has a unique solution which completes the proof. □

Piecewise linear cost and (strict) dissipativity using LP

Using the results of Sec. 2.3 and Sec. 2.4 we state conditions for verifying (strict) dissipativity in the setting considered in this chapter. Consider a *convex*, piecewise defined, linear stage cost $\mathbb{R}^n \to \mathbb{R}$, defined as

$$\ell(x,u) = \max_{i \in \mathcal{I}_{[1,..,L]}} (\ell_{i,x}^\top x + \ell_{i,u}^\top u) \tag{2.29}$$

see Fig. 2.2. Convexity of (2.29) can be verified [5] by rewriting (2.29) as

$$\begin{aligned} \ell(x,u) = \min \quad & t & (2.30) \\ \text{s.t.} \quad & \ell_{i,x}^\top x + \ell_{i,u}^\top u \leq t, \; \forall\, i \in \mathcal{I}_{[1,..,n_L]}. & (2.31) \end{aligned}$$

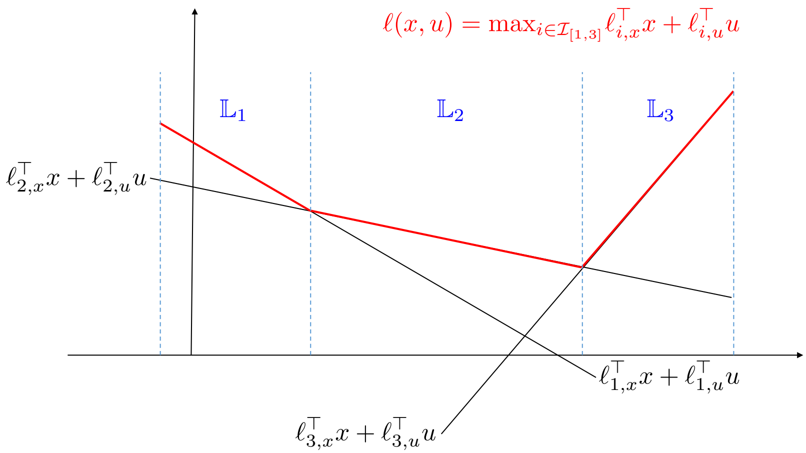


Figure 2.2: Shown is the concept of a piecewise linear defined stage cost as given in (2.29). Black lines are the single linear cost functions. The red polyline indicate the resulting piecewise defined linear cost. The dotted blue segments $\mathbb{L}_i$ visualize the piecewise defined representation (2.32).

Another representation can be obtained [5] by explicitely stating the solution of (2.29) as

$$\ell(x,u) = \begin{cases} \ell_{1,x}^\top x + \ell_{1,u}^\top u, & A_{1,\ell x} x \leq b_{1,\ell x} \ \wedge \ A_{1,\ell u} u \leq b_{1,\ell u} \\ \ell_{2,x}^\top x + \ell_{2,u}^\top u, & A_{2,\ell x} x \leq b_{2,\ell x} \ \wedge \ A_{2,\ell u} u \leq b_{2,\ell u} \\ \vdots & \\ \ell_{n_L,x}^\top x + \ell_{n_L,u}^\top u, & A_{n_L,\ell x} x \leq b_{n_L,\ell x} \ \wedge \ A_{n_L,\ell u} u \leq b_{n_L,\ell u} \end{cases} \tag{2.32}$$

with disjoint polytopic sets[6] $\{(x,u) \in \mathbb{R}^n \times \mathbb{R}^m | A_{i,\ell x} x \leq b_{i,\ell x}, A_{i,\ell u} u \leq b_{i,\ell u}, i \in \mathcal{I}_{[1,n_L]}\}$, see Fig. 2.2. The last representation is also used in the introductory example (1.19). It will play a central role for verification of dissipativity in the case of piecewise defined linear cost. The corresponding

[6]The state space is a polytope.

stage cost (1.10) of the P-step system reads

$$\tilde{\ell}(\tilde{x},\tilde{u}) = \max_{i \in \mathcal{I}_{[1,..,n_{\tilde{L}}]}} \sum_{j=0}^{P-1} \left(\tilde{\ell}_{i,\tilde{x}}^{\top} \tilde{x}_{\tilde{u}}(j,x_{P-1}) + \tilde{\ell}_{i,\tilde{u}}^{\top} \tilde{u}(j) \right) \tag{2.33}$$

and is defined with $n_{\tilde{L}} = n_L^P$ different linear functions[7]. Further (2.33) is also convex like (2.29). For easier reference of the individual cases of (2.33) in the sense of representation (2.32) we introduce the notation

$$\begin{aligned} \tilde{\mathbb{L}}_i : &= \{(Y \in \mathbb{Y} | A_{i,\tilde{\ell}\tilde{x}} \tilde{x}(Y) \leq b_{i,\tilde{\ell}\tilde{x}}, A_{i,\tilde{\ell}\tilde{u}} \tilde{u}(Y) \leq b_{i,\tilde{\ell}\tilde{u}}\} \\ &= \{(Y \in \mathbb{Y} | A_{Y,i} Y \leq b_{Y,i}\}, \text{ with } i \in \mathcal{I}_{[1,n_{\tilde{L}}]}. \end{aligned}$$

Using Cor. 2.3.6 we are able to formulate an explicit criterion for veryfying dissipativity in case of a piecewise defined linear stage cost. For every case of (2.33), i.e. $i = 1,..,n_{\tilde{L}}$ define a local dissipativity equation as

$$\forall Y \in \tilde{\mathbb{L}}_i : \quad 0 \leq q_{0,i}(\tilde{\nu}_i, Y) \tag{2.34}$$

with

$$\begin{aligned} q_{0,i}(\tilde{\nu}_i, Y) =& [\tilde{\nu}_i^{\top}(G - G^{+}) + \tilde{\ell}_{i,x}^{\top} G_{\tilde{x}_{\tilde{\ell}_x}} + \tilde{\ell}_{i,u}^{\top} E] Y \\ &+ \tilde{\nu}_i^{\top}(-\Omega A(P-1)[\Gamma W]_{P-1} - \Omega w_0^*) + \tilde{\ell}_{i,x}^{\top} \Gamma W - \tilde{\ell}_{\Pi}. \end{aligned}$$

Corollary 2.4.5 (Dissipativity in case of convex piecewise defined linear cost)**.** *System* (2.6) *is dissipative with respect to the supply rate* $\tilde{\ell}(\tilde{x},\tilde{u}) - \tilde{\ell}_{\Pi^*}$ *according to the stage cost* (2.29) *and the periodic orbit* Π^* *obtained by minimizing*

$$\begin{aligned} 0 \leq \min_{Y \in \tilde{\mathbb{L}}_i} \quad & (\tilde{\ell}_{i,x}^{\top} G_{\tilde{x}_{\tilde{\ell}_x}} + \tilde{\ell}_{i,u}^{\top} E) Y + \tilde{\ell}_{i,x}^{\top} \Gamma W - \tilde{\ell}_{\Pi} \\ \text{subject to} \quad & (G - G^{+}) Y = \Omega A(P-1)[\Gamma W]_{P-1} + \Omega w_0^* \end{aligned} \tag{2.35}$$

if and only if (2.35) *holds for every* $i \in \mathcal{I}_{[1,n_{\tilde{L}}]}$. ■

Proof. The statement above follows from Cor. 2.3.6. □

[7] For every state-input pair $(\tilde{x}, \tilde{u})$ of the P-step system we have a sum over P stage costs (2.29), see (1.10). Consequently, for each term of the sum there are L cases and because of P terms we have L^P cases in total.

Corollary 2.4.6 (Strict dissipativity for convex piecewise defined linear cost). *Let* $p_i := (\tilde{\ell}_{i,x}^\top G_{\tilde{x}_{\tilde{\ell}_x}} + \tilde{\ell}_{i,u}^\top E)$, $A := (G - G^+)$, $b := \Omega A(P-1)[\Gamma W]_{P-1}$, $C_i := -A_{Y,i}$, *and* $d_i := -b_{Y,i}$ *for every* $i \in \mathcal{I}_{[1,n_{\tilde{L}}]}$. *System* (2.6) *is strictly dissipative with respect to the supply rate* $\tilde{\ell}(\tilde{x},\tilde{u}) - \tilde{\ell}_{\Pi^*}$ *according to the stage cost* (2.29) *and the periodic orbit* Π^* *obtained by minimizing* (2.35) *if and only if the rows of* $[A^\top \; C_{K,i}^\top \; C_{L,i}^\top]$ *are linearly independent for every* $i \in \mathcal{I}_{[1,n_{\tilde{L}}]}$, *and*

$$\begin{aligned} \max_{Y} \quad & \mathbf{1}^\top C_{L,i} Y \\ \text{s.t.} \quad & AY = 0 \\ & C_{K,i} Y = 0 \\ & C_{L,i} Y \geq 0 \end{aligned} \tag{2.36}$$

has a zero maximum for every $i \in \mathcal{I}_{[1,n_{\tilde{L}}]}$. ■

Proof. The statement above follows from Cor. 2.3.7. □

2.5 Example: Simple supply chain network

We study (strict) dissipativity of the system described in 1.3 with respect to the periodic orbit (1.20). In order to apply the criterion for verifying (strict) dissipativity from Sec. 2.4 to system (1.13) with respect to the stage cost (1.19) and optimal orbit candidate (1.20), we begin by fixing the trucks trajectory to be 2-periodic with orbit $(x_{T,P}(2i), x_{T,P}(2i+1)) = (0,1), i \in \mathcal{I}_{\geq 0}$ and inspect the resulting linear periodic time varying system. In this special case we get the following stage cost for the P-step system

$$\tilde{\ell}(\tilde{x},\tilde{u}) = \begin{cases} x_{1,S,1} + 0.5x_{1,S,2} + x_{1,T,L} + u_{0,T,P} + x_{1,R} \\ \quad + x_{0,S,1}^+ + 0.5x_{0,S,2}^+ + x_{0,T,L}^+ + u_{1,T,P} + x_{0,R}^+, \; x_{1,R} \geq 0, x_{0,R}^+ \geq 0 \\ x_{1,S,1} + 0.5x_{1,S,2} + x_{1,T,L} + u_{0,T,P} + x_{1,R} \\ \quad + x_{0,S,1}^+ + 0.5x_{0,S,2}^+ + x_{0,T,L}^+ + u_{1,T,P} - 10x_{0,R}^+, \; x_{1,R} \geq 0, x_{0,R}^+ < 0 \\ x_{1,S,1} + 0.5x_{1,S,2} + x_{1,T,L} + u_{0,T,P} - 10x_{1,R} \\ \quad + x_{0,S,1}^+ + 0.5x_{0,S,2}^+ + x_{0,T,L}^+ + u_{1,T,P} - 10x_{0,R}^+, \; x_{1,R} < 0, x_{0,R}^+ < 0. \end{cases} \tag{2.37}$$

Given the graph trajectory $(0,1)$ for the truck position, the case $x_{0,R} < 0$ and $x_{1,R} \geq 0$ is not possible, because the truck can fill up the retailers storage only at the second time instance. By solving $(P_{\text{orbit,lin}})$ we get the following Lagrange multipliers (storage functions, optimal dual variables) for each case of (2.37):

$$\begin{aligned}
\tilde{v}_1^\top &= [5.7873, 5.1359, 3.8466, -0.41396,\\
&\quad 0.097112, -5.1888, -5.7873, -0.41396], x_{1,R} \geq 0, x_{0,R}^+ \geq 0\\
\tilde{v}_2^\top &= [7.4407, 6.8182, 7.4261, -3.1921,\\
&\quad 0.13043, -6.9947, -7.4407, -3.1921], x_{1,R} \geq 0, x_{0,R}^+ < 0\\
\tilde{v}_3^\top &= [5.7873, 5.1359, 3.8466, -0.41396,\\
&\quad 0.097112, -5.1888, -5.7873, -0.41396], x_{1,R} < 0, x_{0,R}^+ < 0
\end{aligned}$$

with the corresponding optimal values

$$\begin{aligned}
0, x_{1,R} \geq 0, x_{0,R}^+ \geq 0\\
0, x_{1,R} \geq 0, x_{0,R}^+ < 0\\
9, x_{1,R} < 0, x_{0,R}^+ < 0
\end{aligned}$$

which are all greater than zero from which dissipativity for the given trucks position periodic orbit $(0,1)$ follows by Cor. 2.4.5. Further, all values of the LP in Cor. 2.4.6 equal zero, which implies strict dissipativity for the considered truck periodic orbit. It remains to show that all truck position trajectories are worse than the periodic trajectory $(x_{T,P}(2i), x_{T,P}(2i+1)) = (0,1), i \in \mathcal{I}_{\geq 0}$. If this is the case, we conclude by suboptimal operation off the optimal orbit and Rem. 2.2.6, strict dissipativity of (1.20) without the constraint that the truck position must follow the periodic orbit $(x_{T,P}(2i), x_{T,P}(2i+1)) = (0,1), i \in \mathcal{I}_{\geq 0}$. An exact proof for this fact turns out to be non-trivial and is not the main focus of this work. Nevertheless we give the following justification.

- Given a sequence $(\ldots, x_{T,P}(i), x_{T,P}(i+1), x_{T,P}(i+2), x_{T,P}(i+3), \ldots) = (\ldots, *, 0,1,0,1 *, \ldots)$ with $* \in \{0,1\}$, the periodic orbit (1.20) is our best possible operation guess for $x(i), x(i+1)$ because it is always optimal to have as little goods in the supply chain as possible such that the retailer storage remains positive.

- There will always be just finite constant subsequences $(x_{T,P}(i), x_{T,P}(i+1), \ldots) = (1,1,\ldots)$ and $(x_{T,P}(i), x_{T,P}(i+1), \ldots) = (0,0,\ldots)$, because otherwise the retail store storage value will decrease by the nominal demand until it is infeasible. This is because the retailer and truck storage is finite and therefore at some point the truck must travel between the supplier and the retailer in order to preserve feasibility w.r.t. the state constraints.

- Consider that the truck remains at the supplier for $\tau + 1$ time instances with $\tau \in \mathcal{I}_{\geq 0}$ and therefore leaves the periodic $(0{,}1)$ orbit in case $\tau > 0$. For such a scenario, consider the truck's position orbit given in Tab. 2.1. The remaining states and inputs are chosen to minimize the overall average cost until we reach the periodic orbit $(0{,}1)$ again. The average cost for the trajectory shown in Tab. 2.1 can be calculated using the Gauss sum and equals

$$\eta_1(\tau) = \frac{\tau^2 + 7\tau + 28}{2\tau + 8}. \tag{2.38}$$

  For $\tau = 0$, the truck remains for one time instance at the supplier, and the corresponding average cost is $\eta_1(0) = 3.5$ which equals the average cost of (1.20). For all $\tau > 0$ we have an average cost $\eta_1(\tau) > 3.5$. Therefore, for the considered scenario it is suboptimal to leave the truck position periodic orbit $(0{,}1)$.

- Analogously, in Tab. 2.2 the truck remains for $\tau + 1$ time instances at the retailer. The average optimal cost w.r.t. τ reads

$$\eta_2(\tau) = \frac{\tau^2 + 7\tau + 14}{2\tau + 4}. \tag{2.39}$$

  Again, $\eta_2(0) = 3.5$ which equals the optimal average cost of (1.20) and for all $\tau > 0$ we have $\eta(\tau) > 3.5$. Consequently in this scenario it is suboptimal to leave the truck position orbit $(0{,}1)$ also.

The justification above is by no means a complete proof. As a consequence, we have to rely on 'expert knowledge'. The following control methods however rely on the strict dissipativity property. Therefore, if the closed loop shows a different behavior than expected, we can fix the truck's position to $(0{,}1)$, for which we have rigorously proven strict dissipativity.

	$k=1$	$k=2$	$k=3$	..	$k=\tau+3$	$k=\tau+4$
$x_{S,1}(k)$	$\tau+2$	0	0	0 .. 0	2	0
$x_{S,2}(k)$	0	0	0	0 .. 0	0	0
$x_{T,P}(k)$	0	1	0	0 .. 0	0	1
$x_{T,L}(k)$	0	$\tau+2$	0	0 ... 0	0	2
$x_R(k)$	1	0	$\tau+1$	τ .. 2	1	0
$u_{T,P}(k)$	1	1	0	0 .. 0	1	1
$u_{T,L}(k)$	$\tau+2$	$-\tau-2$	0	0 .. 0	0	0
$u_S(k)$	0	0	0	0 .. 2	0	0

Table 2.1: Optimal state and input trajectory in case that the truck remains at the supplier for $\tau+1$ time instances. The corresponding cost is given in (2.38).

	$k=1$	$k=2$	$k=3$	..	$k=\tau+2$
$x_{S,1}(k)$	$\tau+2$	0	0	0 ... 0	0
$x_{S,2}(k)$	0	0	0	0 ... 0	0
$x_{T,P}(k)$	0	1	1	1 ... 1	1
$x_{T,L}(k)$	0	$\tau+2$	0	0 ... 0	0
$x_R(k)$	1	0	$\tau+1$	τ ... 2	2
$u_{T,P}(k)$	1	0 (1)	0	0 ... 0	1
$u_{T,L}(k)$	$\tau+2$	$-\tau-2$	0	0 ... 0	0
$u_S(k)$	0	0	0	0 ... 0	0

Table 2.2: Optimal state and input trajectory in case that the truck remains at the retailer for τ time instances. If $\tau=0$ we must choose $u_{T,P}=1$ at $k=2$ for optimality. The corresponding cost is given in (2.39).

3 Economic MPC for optimal periodic operation

3.1 Introduction

Economic model predictive control (EMPC) schemes have been studied extensively in recent years. They differ from classical stabilizing model predictive control schemes in terms of their general performance objective function which does not need to be chosen such that the controller stabilizes the system state with respect to an a-priori given reference point or trajectory. Such references are typically selected through an auxiliary optimization or process expertise, in order to achieve good overall closed loop system performance with respect to a certain general (economic) performance objective. In contrast, as the name suggests, the economic objective is directly used in economic model predictive control schemes. Loosely speaking, given a small disturbance pushing the system away from optimal operation, the advantage is that the system will be economically controlled towards optimal operation compared to classical MPC which would push the system to the reference most of the time regardless of the true economic objective.

However, since the cost can be chosen arbitrarily, the closed loop system has potentially chaotic behavior. In particular it will not necessarily converge to a steady-state. In the literature, various EMPC schemes have been developed, that ensure certain performance, e.g. [1,3]. In addition criterions are provided which allow to determine whether or not for example steady-state operation is optimal and if the closed loop system will converge to steady-state [19]. It is not only interesting from a theoretical point of view to determine a-priori how the closed loop will behave or how we have to design the EMPC scheme such that for example steady-state operation is optimal. In practice, especially in safety critical processes that are under strict observation, e.g. chaotic behavior can hardly be monitored.

Despite optimal steady-state operation, in nature, economics, and engineering applications, periodic operation plays an essential role, e.g. in case of evolutionary processes, sleeping rhythms, human walk, engines, parking a car sideways, or supply chain networks as considered in this work as application. More general, in terms of controllability analysis of non-linear systems, Lie-brackets play a central role. The basic idea is that certain non-linear systems can only be controlled by using some kind of periodic input pattern, see e.g. [6].

Our contribution in this section is a novel economic model predictive control scheme for optimal periodic operation, based on ideas from [1] for the steady-state case. The algorithm we propose is shown to be recursively feasible and has an average performance which is no worse than that of optimal periodic operation. Further in case of strict dissipativity, our control scheme is proven to asymptotically stabilize the optimal periodic orbit.

Related work

In [18] an EMPC scheme without terminal constraints is presented, which does not require a terminal cost or a terminal constraint. Based on controllability assumptions and strict dissipativity, they provide a bound on suboptimality w.r.t. closed loop asymptotic average performance and practical convergence guarantees to the optimal periodic orbit. The main advantage is the resulting, simple online optimization problem. However, since the performance and convergence bounds are related to the planning horizon, in some cases the latter has to be chosen quite large in order to achieve nearly optimal performance. We showed this effect using the simple supply chain example (Sec. 1.3) as well as in the application section of this work using a complex supply chain network.

There also exists a method, more closely related to our approach [24]. It is based on a terminal region and terminal cost and thus generalizes [23]. Compared to their assumptions, we propose a control scheme that is also applicable in case of non-controllability in a local region around the optimal periodic orbit as long as the system is controllable over one period. As we will see in the application section, it would not be possible to apply the method presented in [24] to the example of supply chain networks considered in this work. Furthermore, our assumptions on the terminal controller and terminal cost can be treated similar to the case when steady-state operation is optimal, as e.g. investigated in [1]. The strict dissipativity assumptions they introduce are time-varying, P-periodic, which differs from common literature, as e.g. [19] and are more involved, even in a linear setting. Further, they do not provide closed loop performance guarantees.

Table 3.1: Notation used in Chap. 3 with $k \in \mathcal{I}_{\geq t}$, if not specified otherwise.

Symbol	Definition
$\boldsymbol{u}$	$\boldsymbol{u} \in \mathbb{U}^N$
$u(k)$	See above: elements of $\boldsymbol{u}$ for $k \in \mathcal{I}_{[0,N-1]}$
$\boldsymbol{u}^*(t)$	$\boldsymbol{u}^*(t) \in \mathbb{U}^N$, a solution of $(P_{\text{EMPC-P}})$ with initial condition $x(t)$
$u^*(k)$	See above: elements of $\boldsymbol{u}^*(t)$ for $k \in \mathcal{I}_{[t,t+N-1]}$
$\bar{u}^*(k)$	Candidate input signal as defined in (3.3)
$\bar{\boldsymbol{u}}_{\boldsymbol{u}^*(t)}(k)$	$\bar{\boldsymbol{u}}_{\boldsymbol{u}^*(t)}(k) = (\bar{u}^*(k),\ldots,\bar{u}^*(k+N-1))$, a tuple of candidate inputs (3.4)
$u_{\text{MPC}}(t)$	Inputs applied to system (1.5) when using an MPC algorithm starting with initial condition $x(0)$
$\tilde{u}(k)$	$\tilde{u}(k) = (u(k),..,u(k+P-1))$, $k \in \mathcal{I}_{[0,N-P]}$
$\tilde{u}^*(k)$	$\tilde{u}^*(k) = (u^*(k),..,u^*(k+P-1))$, $k \in \mathcal{I}_{[t,t+N-P]}$
$\tilde{\bar{u}}^*(k)$	$\tilde{\bar{u}}^*(k) = (\bar{u}^*(k),..,\bar{u}^*(k+P-1))$, $k \in \mathcal{I}_{[t,t+N-P]}$
x	Initial condition for $(P_{\text{EMPC-P}})$
$x_{\boldsymbol{u}}(k, x)$	Trajectory of system (1.5) at time $k \in \mathcal{I}_{[0,N]}$, starting with initial condition x and applying $\boldsymbol{u}$
$x_{\boldsymbol{u}^*(t)}(k, x(t))$	Trajectory of system (1.5) at time $k \in \mathcal{I}_{[t,t+N]}$, starting with initial condition $x(t)$ and applying $\boldsymbol{u}^*(t)$
$x_{\text{MPC}}(t)$	Closed loop trajectory of system (1.5) under application of an MPC algorithm, starting at $x(0)$
$x_{\text{cMPC},\boldsymbol{u}^*(t)}(k)$	Closed loop trajectory of system (1.5) under candidate inputs $\bar{u}^*(k)$ starting from $x_{\text{MPC}}(t)$
$\tilde{x}_{\boldsymbol{u}}(k,x)$	$\tilde{x}_{\boldsymbol{u}}(k,x) = (x_{\boldsymbol{u}}(k-P+1,x),.., x_{\boldsymbol{u}}(k,x))$ and filled with zero elements in case $k-P+1<0$
$\tilde{x}_{\boldsymbol{u}^*(t)}(k,x(t))$	$\tilde{x}_{\boldsymbol{u}^*(t)}(k,x(t)) =$ $(x_{\boldsymbol{u}^*(t)}(k-P+1, x(t)),..,x_{\boldsymbol{u}^*(t)}(k,x(t))$ and filled with zero elements in case $k-P+1<t$
$\tilde{x}_{\text{MPC}}(t)$	$\tilde{x}_{\text{MPC}}(t) = (x_{\text{MPC}}(t-P+1),..,x_{\text{MPC}}(t))$ and filled with zero elements in case $k-P+1<t$
$\tilde{x}_{\text{cMPC},\boldsymbol{u}^*(t)}(k)$	$\tilde{x}_{\text{cMPC},\boldsymbol{u}^*(t)}(k) =$ $(x_{\text{cMPC},\boldsymbol{u}^*(t)}(k-P+1),..,x_{\text{cMPC},\boldsymbol{u}^*(t)}(k))$ and filled with zero elements in case $k-P+1<t$

3.2 Assumptions and algorithm

We consider systems as introduced in Sec. 1.2 *without* disturbances, i.e. w.l.o.g. $w(k) = 0$ for all $k \in \mathcal{I}_{\geq 0}$. We use the notation $(\tilde{x})_{P-1} = x_{P-1}$.

Assumption 3.2.1 (Terminal controller, set and cost). *Let $(x_i^p, u_i^p) \in \Pi$ for $i \in \mathcal{I}_{[0,P-1]}$. There exists a compact $\mathbb{X}_f \subseteq \mathbb{X}$ such that for all $i \in \mathcal{I}_{[0,P-1]}$ the set of phase shifted orbits $\tilde{\Pi}_{\mathbb{X}}$ of $\Pi_{\mathbb{X}}$ is contained in $\mathbb{X}_f^P$. Further assume that there exists a feedback law $\tilde{\kappa}_f : \mathbb{X}^P \to \mathbb{U}^P$ and a continuous terminal cost $V_f : \mathbb{X}_f \to \mathbb{R}$ such that $\forall \tilde{x}$ with $x_{P-1} \in \mathbb{X}_f$:*

1. $\tilde{\kappa}_f(\tilde{x}) \in \mathbb{U}^P$ $\quad\triangleright$ *feasibility*

2. $f^P(\tilde{x}, \tilde{\kappa}_f(\tilde{x}), 0) \in \mathbb{X}_f^P$ $\quad\triangleright$ *positive invariance of* $\mathbb{X}_f^P$

3. $$V_f((f^P(\tilde{x}, \tilde{\kappa}_f(\tilde{x}), 0))_{P-1}) - V_f((\tilde{x})_{P-1}) \leq -\tilde{\ell}(\tilde{x}, \tilde{\kappa}_f(\tilde{x})) + \sum_{i=0}^{P-1} \ell(x_i^p, u_i^p).$$

Without loss of generality let $V_f(x) \geq 0 \; \forall x \in \mathbb{X}_f$. ■

Ass. 3.2.1 can be interpreted as the common terminal region stability assumption in economic model predictive control [1] with respect to the P-step system. Using this assumption we will show asymptotic stability of $\tilde{\Pi}_{\mathbb{X}}$ which corresponds to the classical steady state stability in terms of the P-step system.

Remark 3.2.2 (Construction of terminal cost function). *By defining the dynamics $\bar{f}(x, \tilde{u}) := f(f(..f(x, u_0, 0), .., u_{P-2}, 0), u_{P-1}, 0)$ and stage cost $\bar{\ell}(x, \tilde{u}) = \tilde{\ell}(\tilde{x}, \tilde{u})$ with $\tilde{x} = (*, *, .., x)$, $* \in \mathbb{R}^n$ (arbitrary), we can use the method described in [1] in order to construct V_f.* ■

Let $N = N_1 P$ with $N_1 \in \mathcal{I}_{>0}$[1]. Define the open loop optimization problem

$$(P_{\text{EMPC-P}}) \begin{cases} \min_{\boldsymbol{u}\in\mathbb{U}^N} J_{\mathrm{MPC}}(x,\boldsymbol{u}) \\ \text{s.t. for all } k \in \mathcal{I}_{[0,N-1]} : \\ \qquad x_{\boldsymbol{u}}(k+1,x) = f(x_{\boldsymbol{u}}(k,x),u(k),0) \\ \qquad x_{\boldsymbol{u}}(k,x) \in \mathbb{X} \\ \qquad u(k) \in \mathbb{U} \\ \qquad x_{\boldsymbol{u}}(N,x) \in \mathbb{X}_f \\ \qquad x_{\boldsymbol{u}}(0,x) = x \end{cases}$$

with finite time open loop cost functional

$$\begin{aligned} J_{\mathrm{MPC}}(x,\boldsymbol{u}) :&= \sum_{k=0}^{N-1} \ell(x_{\boldsymbol{u}}(k,x),u(k)) + V_f(x_{\boldsymbol{u}}(N,x)) \\ &= \sum_{k=0}^{N/P-1} \tilde{\ell}(\tilde{x}_{\boldsymbol{u}}(kP,x),\tilde{u}(kP)) + V_f(x_{\boldsymbol{u}}(N,x)) \end{aligned} \tag{3.1}$$

which will be solved for $\boldsymbol{u} = (u(0), u(1),..,u(N-1))$ at each time step $tP \in \mathcal{I}_{\geq 0}$ using the current system state $x = x(tP)$. Let the optimal input sequence of $(P_{\text{EMPC-P}})$ at time t be denoted by $\boldsymbol{u}^*(t) = (u^*(t),..,u^*(t+N-1))$ with corresponding optimal states $x_{\boldsymbol{u}^*(t)}(k, x(t))$ for $k \in \mathcal{I}_{[t,t+N]}$ and $x_{\boldsymbol{u}^*(t)}(t, x(t)) = x(t)$, see also Tab. 3.1. W.l.o.g. assume that the value of J_{MPC} will be greater or equal than zero. This is valid, since w.l.o.g. we can assume that ℓ and V_f are always greater or equal than zero due to compactness of the respective sets and continuity of the functions.

Assumption 3.2.3. *The optimization problem* $(P_{\text{EMPC-P}})$ *is feasible at time* $t = 0$ *for* $x = x(0)$. ■

In Alg. 1 we propose the P-step economic model predictive control algorithm for optimal periodic operation. If Alg. 1 is applied, starting with initial state $x(0)$, we denote the closed loop system states simply by $x_{\text{MPC}}(t)$ and the applied inputs by $u_{\text{MPC}}(t)$, $t \in \mathcal{I}_{\geq 0}$ with corresponding definitions for the P-step system representation, see Tab. 3.1.

[1]This means we only consider multiples of the period length P for the planning horizon.

3.3 Recursive feasibility

Theorem 3.3.1 (Recursive feasibility of ($P_{\text{EMPC-P}}$)). *Let Ass. 3.2.1 and Ass. 3.2.3 hold. Then Alg. 1 is recursively feasible.* ■

Proof. We prove recursive feasibility as it is done typically [20] by breaking down the P-step system control law $\tilde{\kappa}_f$ in terms of single time steps rather than considering the P-step system steps. Note, that we can rewrite $\tilde{\kappa}_f(\tilde{x})$ from Ass. 3.2.1 as

$$\tilde{\kappa}_f(\tilde{x}) = (\kappa_{f,0}(\tilde{x}), \kappa_{f,1}(\tilde{x}), ..., \kappa_{f,P-1}(\tilde{x})) \tag{3.2}$$

by the definition of the P-step system. By Ass. 3.2.3, Alg. 1 is feasible at $t = 0$. Consider the candidate input solution starting at time $t + \tau$ for $\tau \in \mathcal{I}_{\geq 1}, k \in \mathcal{I}_{[t+\tau,t+\tau+N-1]}$

$$\bar{u}^*(k) := \begin{cases} u^*(k), & k \in \mathcal{I}_{[t,t+N-1]} \\ \tilde{\kappa}_{f,(k-t)\mathrm{mod}P}\left(\tilde{x}_{\mathrm{cMPC},\boldsymbol{u}^*(t)}(P\lfloor \frac{k-t}{P} \rfloor + t)\right), \text{else} \end{cases} \tag{3.3}$$

specified by using the notation (3.2). We denote the closed loop 'candidate' states by $x_{\mathrm{cMPC},\boldsymbol{u}^*(t)}(k)$ with $x_{\mathrm{cMPC},\boldsymbol{u}^*(t)}(t) = x_{\mathrm{MPC}}(t)$ if we use $\boldsymbol{u}^*(t)$ to construct the applied candidate inputs $\bar{u}^*(k)$, $k \in \mathcal{I}_{\geq t}$. The corresponding notations for the P-step system representation is given in Tab. 3.1. We define

$$\bar{\boldsymbol{u}}_{\boldsymbol{u}^*(t)}(k) = (\bar{u}^*(k), \bar{u}^*(k+1), ..., \bar{u}^*(k+N-1)) \tag{3.4}$$

as the corresponding candidate input trajectory, which is a feasible periodic completion of the nominal input trajectory $\boldsymbol{u}^*(t)$. Therefore the index $(k-t)$ $\mathrm{mod}\ P$ 'chooses' the right successor input element of the nominal terminal feedback law (3.2) for the system. The last state considered in the open loop optimization is $x_{\boldsymbol{u}^*(t)}(t+N)$. For $k = t+N$ and e.g. $\tau = 1$ we then have

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Algorithm 1 Economic model predictive control for optimal periodic operation

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1: **procedure** EMPC-P(initial state $x(0)$)
2: **for** $k = 0,1,...$ **do**
3: solve ($P_{\text{EMPC-P}}$) with initial condition $x_{\mathrm{MPC}}(kP)$
4: apply the first P inputs of $\boldsymbol{u}^*(kP)$ to the system (1.5)
5: **end for**
6: **end procedure**

$(N) \mod P = 0$, because $N = N_1 P$. The P-step feedback (3.2) is based on the last P-step system state. I.e. $P\lfloor \frac{k-t}{P} \rfloor + t$ is constant within one period and increments once a period is completed. Again for e.g. $k = t + N$ we get $\kappa_{f,0}(\tilde{x}_{\boldsymbol{u}^*(t)}(t+N))$. By the definition of the P-step system according to system (1.5) and Ass. 3.2.1 it follows directly that (3.4) is a feasible input trajectory for all $\tau \in \mathcal{I}_{\geq 1}$ which completes the proof. □

3.4 Asymptotic average performance

Theorem 3.4.1 (Asymptotic average performance)**.** *If Ass. 3.2.1 and Ass. 3.2.3 hold, then under application of Alg. 1 the closed loop system has an average performance which is no worse than that of the optimal periodic orbit* $\{P, \Pi\}$*, i.e.*

$$\frac{1}{P}\sum_{k=0}^{P-1} \ell(x_k^p, u_k^p) \geq \limsup_{T\to\infty} \frac{\sum_{k=0}^{T-1} \ell(x_{\mathrm{MPC}}(k), u_{\mathrm{MPC}}(k))}{T} \tag{3.5}$$

with $(x_i^p, u_i^p) \in \Pi$ *for* $i \in \mathcal{I}_{[0,P-1]}$. ■

Proof. Consider $J_{\mathrm{MPC}}(x_{\mathrm{cMPC},\boldsymbol{u}^*(0)}(t), \bar{\boldsymbol{u}}_{\boldsymbol{u}^*(0)}(t))$ which is the open loop finite time cost at time t by using the candidate input defined in (3.4). We have

$$\begin{aligned} J_{\mathrm{MPC}}(x_{\mathrm{cMPC},\boldsymbol{u}^*(0)}(t), \bar{\boldsymbol{u}}_{\boldsymbol{u}^*(0)}(t)) = &\sum_{k=t+P}^{t+N-1} \ell(x_{\mathrm{cMPC},\boldsymbol{u}^*(0)}(k), \bar{u}^*(k)) \\ &+ \tilde{\ell}(\tilde{x}_{\mathrm{cMPC},\boldsymbol{u}^*(0)}(t), \tilde{\bar{u}}^*(t))) \\ &+ V_f(x_{\mathrm{cMPC},\boldsymbol{u}^*(0)}(t+N)) \end{aligned}$$

which corresponds to the optimal open loop, finite time optimal cost due to the definition of the candidate input and

$$\begin{aligned} &J_{\mathrm{MPC}}(x_{\mathrm{cMPC},\boldsymbol{u}^*(0)}(t+P), \bar{\boldsymbol{u}}_{\boldsymbol{u}^*(0)}(t+P)) = \\ &\qquad \sum_{k=t+P}^{t+N-1} \ell(x_{\mathrm{cMPC},\boldsymbol{u}^*(0)}(k), \bar{u}^*(k)) \\ &\qquad + \tilde{\ell}(x_{\mathrm{cMPC},\boldsymbol{u}^*(0)}(t+N), \tilde{\kappa}_f(\tilde{x}_{\mathrm{cMPC},\boldsymbol{u}^*(0)}(t+N))) \\ &\qquad + V_f(x_{\mathrm{cMPC},\boldsymbol{u}^*(0)}(t+N+P)). \end{aligned}$$

In the following we use the short notations

$$\tilde{J}_{\mathrm{cMPC}}(t) := J_{\mathrm{MPC}}(x_{\mathrm{cMPC},\boldsymbol{u}^*(0)}(t), \bar{\boldsymbol{u}}_{\boldsymbol{u}^*(0)}(t)) \tag{3.6}$$

and

$$\tilde{J}_{\mathrm{cMPC}}(t+P) := J_{\mathrm{MPC}}(x_{\mathrm{cMPC},\boldsymbol{u}^*(0)}(t+P), \bar{\boldsymbol{u}}_{\boldsymbol{u}^*(0)}(t+P)). \tag{3.7}$$

We have

$$\begin{aligned}
\tilde{J}_{\mathrm{cMPC}}(t+P) - \tilde{J}_{\mathrm{cMPC}}(t) = \\
& - \tilde{\ell}(\tilde{x}_{\mathrm{cMPC},\boldsymbol{u}^*(0)}(t), \tilde{\bar{u}}^*(t))) \\
& + V_f(x_{\mathrm{cMPC},\boldsymbol{u}^*(0)}(t+N+P)) \\
& - V_f(x_{\mathrm{cMPC},\boldsymbol{u}^*(0)}(t+N)) \\
& + \tilde{\ell}(\tilde{x}_{\mathrm{cMPC},\boldsymbol{u}^*(0)}(t+N), \tilde{\kappa}_f(\tilde{x}_{\mathrm{cMPC},\boldsymbol{u}^*(0)}(t+N)))
\end{aligned}$$

The last three summands are upper bounded via Ass. 3.2.1 and we get

$$\tilde{J}_{\mathrm{cMPC}}(t+P) - \tilde{J}_{\mathrm{cMPC}}(t) \leq -\tilde{\ell}(\tilde{x}_{\mathrm{cMPC},\boldsymbol{u}^*(0)}(t), \tilde{\bar{u}}^*(t))) + \sum_{i=0}^{P-1} \ell(x_i^p, u_i^p). \tag{3.8}$$

From here we can proceed analogously to [1] by taking the average on both sides of (3.8) in terms of the period length P and calculate

$$\begin{aligned}
&\liminf_{T\to\infty} \frac{\sum_{k=t}^{T} \left(\tilde{J}_{\mathrm{cMPC}}(kP+P) - \tilde{J}_{\mathrm{cMPC}}(kP) \right)}{TP} \leq \\
&\quad \liminf_{T\to\infty} \frac{\sum_{k=t}^{T} \left(-\tilde{\ell}(\tilde{x}_{\mathrm{cMPC},\boldsymbol{u}^*(0)}(kP), \tilde{\bar{u}}^*(kP))) + \sum_{i=0}^{P-1} \ell^{\mathrm{int}}(z_i^p, v_i^p) \right)}{TP}.
\end{aligned} \tag{3.9}$$

By having a closer look at the LHS of (3.9) we notice the telescoping series and get

$$\begin{aligned}
\liminf_{T\to\infty} \frac{\tilde{J}_{\mathrm{cMPC}}(TP+P) - \tilde{J}_{\mathrm{cMPC}}(t)}{TP} &= \liminf_{T\to\infty} \left(\underbrace{\frac{\tilde{J}_{\mathrm{cMPC}}(TP+P)}{TP}}_{\geq 0} + \underbrace{\frac{\tilde{J}_{\mathrm{cMPC}}(t)}{TP}}_{\to 0} \right) \\
&\geq 0.
\end{aligned}$$

We conclude

$$\underbrace{(..)}_{\geq 0} + \limsup_{T\to\infty} \frac{\sum_{k=t}^{T}\left(\tilde{\ell}(\tilde{x}_{\text{cMPC},\boldsymbol{u}^*(0)}(kP),\tilde{\tilde{u}}^*(kP)))\right)}{TP}$$
$$\leq \liminf_{T\to\infty} \frac{\sum_{k=t}^{T}\left(\sum_{i=0}^{P-1}\ell(x_i^p,u_i^p)\right)}{TP}$$
$$\leq \frac{1}{P}\sum_{i=0}^{P-1}\ell(x_i^p,u_i^p)$$

which is by the definition of the P-step system stage cost (1.10) equal to (3.5). By suboptimality of the candidate solution we can always choose the optimal open loop, finite time value function of ($P_{\text{EMPC-P}}$) instead of $J_{\text{aux}}(t+P)$ in (3.8) which therefore implies optimal operation using Alg. 1 and thus completes the proof. □

3.5 Asymptotic stability of the optimal periodic orbit

In this section we investigate stability by constructing a Lyapunov function. We introduce a Lyapunov function which is different to the one from [1], however the concept of rotated cost functionals is used.

Assumption 3.5.1. *System (1.5) is strictly dissipative with respect to Π according to Def. 2.2.4 with a continuous storage function $\tilde{\lambda}$.* ■

Assumption 3.5.2. *For all $x(0) \in \Pi_{\mathbb{X}}$ the solution $\boldsymbol{u}^*$ of ($P_{\text{EMPC-P}}$) is element of $\tilde{\Pi}_{\mathbb{U}}^{N/P}$ such that for the corresponding states it holds $\tilde{x}_{\boldsymbol{u}^*}(k) \in \tilde{\Pi}_{\mathbb{X}}$ for all $k \in \mathcal{I}_{[P-1,N]}$.* ■

Remark 3.5.3. *Ass. 3.5.2 e.g. fulfilled when using terminal equality constraints. Otherwise we have to verify the condition in Ass. 3.5.2 for every element of[2] $\Pi_{\mathbb{X}}$. In case that Ass. 3.5.2 is not valid, there exists a state $\bar{x} \in \Pi_{\mathbb{X}}$ such that if $\bar{x}$ is the initial condition we will not stay necessarily on Π under application of Alg. 1. Without rigorous analysis we expect that in this case, the closed loop system will switch the phase on which it follows the orbit $\Pi_{\mathbb{X}}$.* ■

[2]Solve ($P_{\text{EMPC-P}}$) for every element $x(0) \in \Pi_{\mathbb{X}}$ and verify if $\boldsymbol{u}^* \in \tilde{\Pi}_{\mathbb{U}}^{N/P}$.

Rotated cost functionals

Let $\tilde{\ell}_\Pi$ be the sum of stage costs along the periodic orbit Π. The *rotated stage cost* is defined as

$$\tilde{L}(\tilde{x},\tilde{u}) := \tilde{\ell}(\tilde{x},\tilde{u}) - \tilde{\ell}_\Pi + \tilde{\lambda}(\tilde{x}) - \tilde{\lambda}(f^P(\tilde{x},\tilde{u},0)), \tag{3.10}$$

by the definition of the P-step system stage cost function (1.10). Let the *rotated P-step terminal cost* $\{\tilde{x} \in \mathbb{X}^P | x_{P-1} \in \mathbb{X}_f\} \to \mathbb{R}$ be

$$\tilde{V}_f(\tilde{x}) := V_f((\tilde{x})_{P-1}) + \tilde{\lambda}(\tilde{x}),$$

and the *auxiliary objective*

$$J_{\mathrm{aux}}(x(t),\boldsymbol{u}) := \sum_{k=0}^{N/P-1} \tilde{L}(\tilde{x}_{\boldsymbol{u}}(t+kP),\tilde{u}(t+kP)) + \tilde{V}_f(\tilde{x}_{\boldsymbol{u}}(t+N)). \tag{3.11}$$

Lemma 3.5.4. *The rotated objective function yields the same minimizer as J_{MPC}, i.e.*

$$\operatorname*{argmin}_{\boldsymbol{u}\in\mathbb{U}^N(x(t))} J_{\mathrm{MPC}}(x(t),\boldsymbol{u}) = \operatorname*{argmin}_{\boldsymbol{u}\in\mathbb{U}^N(x(t))} J_{\mathrm{aux}}(x(t),\boldsymbol{u}).$$

■

Proof. We have

$$\begin{aligned}
J_{\mathrm{aux}}(x(t),\boldsymbol{u}) = &\sum_{k=0}^{N/P-1} \Big(\tilde{\ell}(\tilde{x}_{\boldsymbol{u}}(t+kP),\tilde{u}(t+kP)) - \tilde{\ell}_\Pi\Big) \\
&+ \sum_{k=0}^{N/P-1} \Bigg[\underbrace{\tilde{\lambda}(\tilde{x}_{\boldsymbol{u}}(t+kP)) - \tilde{\lambda}(f^P(\tilde{x}_{\boldsymbol{u}}(t+kP),\tilde{u}(t+kP),0))}_{\text{telescopic w.r.t. } k}\Bigg] \\
&+ V_f(x_{\boldsymbol{u}}(t+N)) + \tilde{\lambda}(\tilde{x}_{\boldsymbol{u}}(t+N)) \\
= &\underbrace{\sum_{k=0}^{N/P-1} \tilde{\ell}(\tilde{x}_{\boldsymbol{u}}(t+kP),\tilde{u}(t+kP)) + V_f(x_{\boldsymbol{u}}(t+N))}_{=J_{\mathrm{MPC}}(x(t),\boldsymbol{u})} \\
&\underbrace{- \sum_{k=0}^{N/P-1} \tilde{\ell}_\Pi + \tilde{\lambda}(\tilde{x}_{\boldsymbol{u}}(t))}_{\text{constant}}.
\end{aligned}$$

Consequently we have $J_{\mathrm{aux}}(x(t),\boldsymbol{u}) = J_{\mathrm{MPC}}(x(t),\boldsymbol{u}) + \text{const.}$, and therefore J_{aux} and J_{MPC} have the same minimizer. □

Stability analysis

Consider the set

$$X_N := \{x \in \mathbb{X} | \exists \boldsymbol{u} \in \mathbb{U}^N(x) \text{ s.t. } x_{\boldsymbol{u}}(t+N,x) \in \mathbb{X}_f\} \tag{3.12}$$

for which ($P_{\text{EMPC-P}}$) is feasible and

$$\tilde{X}_N := \{\tilde{x} \in \mathbb{X}^P | x_{P-1} \in X_N\}.$$

Let $V : \tilde{X}_N \to \mathbb{R}$ be defined as

$$\begin{aligned} V(\tilde{x}) = -\sum_{k=0}^{\infty} [& J_{\mathrm{aux}}(x_{\mathrm{cMPC},\boldsymbol{u}^*(0)}(kP+P),\bar{\boldsymbol{u}}_{\boldsymbol{u}^*(0)}(kP+P)) \\ & - J_{\mathrm{aux}}(x_{\mathrm{cMPC},\boldsymbol{u}^*(0)}(kP),\bar{\boldsymbol{u}}_{\boldsymbol{u}^*(0)}(kP))] \end{aligned} \tag{3.13}$$

with $\boldsymbol{u}^*(0)$ the solution to ($P_{\text{EMPC-P}}$) for initial condition x_{P-1} and $\bar{u}^*(.)$ the corresponding candidate solution, see Tab. 3.1. The function $V(\tilde{x})$ is the infinite sum over (finite time) open loop auxiliary cost differences along the closed loop trajectory resulting from applying the candidate input sequence. In the remainder of this chapter we prove that (3.13) is a Lyapunov function for the P-step system which will finally be used for the stability theorem.

Lemma 3.5.5. *For all $\tilde{x} \in \{\tilde{x} \in \mathbb{X}^P | x_{P-1} \in \mathbb{X}_f\}$ it holds*

$$\tilde{V}_f(f^P(\tilde{x}, \tilde{\kappa}_f(\tilde{x}),0)) - \tilde{V}_f(\tilde{x}) \leq -\tilde{L}(\tilde{x}, \tilde{\kappa}_f(\tilde{x})). \tag{3.14}$$

■

Proof. By Ass. 3.2.1 for $\tilde{x} \in \{\tilde{x} \in \mathbb{X}^P | x_{P-1} \in \mathbb{X}_f\}$ it holds

$$V_f((f^P(\tilde{x}, \tilde{\kappa}_f(\tilde{x}),0))_{P-1}) - V_f((\tilde{x})_{P-1}) \leq -\tilde{\ell}(\tilde{x}, \tilde{\kappa}_f(\tilde{x})) + \tilde{\ell}_{\Pi}.$$

Adding $-\tilde{\lambda}(\tilde{x}) + \tilde{\lambda}(f^P(\tilde{x}, \tilde{\kappa}_f(\tilde{x}),0))$ on both sides yields

$$\begin{aligned} V_f((f^P(\tilde{x}, \tilde{\kappa}_f(\tilde{x}),0))_{P-1}) - V_f((\tilde{x})_{P-1}) - \tilde{\lambda}(\tilde{x}) + \tilde{\lambda}(f^P(\tilde{x}, \tilde{\kappa}_f(\tilde{x}),0)) \\ \leq -\tilde{L}(\tilde{x},\tilde{\kappa}_f(\tilde{x})) \end{aligned}$$

whereas the LHS equals $\tilde{V}_f(f^P(\tilde{x}, \tilde{\kappa}_f(\tilde{x}),0)) - \tilde{V}_f(\tilde{x})$ and thus we have shown (3.14). □

Lemma 3.5.6. *For all $x(0) \in X_N$ it holds for all $k \in \mathcal{I}_{\geq 0}$*

$$\begin{aligned} &J_{\text{aux}}(x_{\text{cMPC},\boldsymbol{u}^*(0)}(kP+P), \bar{\boldsymbol{u}}_{\boldsymbol{u}^*(0)}(kP+P)) \\ &\qquad - J_{\text{aux}}(x_{\text{cMPC},\boldsymbol{u}^*(0)}(kP), \bar{\boldsymbol{u}}_{\boldsymbol{u}^*(0)}(kP+P)) \\ &\leq -\alpha\left(|(\tilde{x}_{\text{cMPC},\boldsymbol{u}^*(0)}(kP), \tilde{\bar{u}}^*(kP))|_{\Pi}\right). \end{aligned}$$

■

Proof. Consider for all $k \in \mathcal{I}_{\geq 0}$

$$\begin{aligned} &J_{\text{aux}}(x_{\text{cMPC},\boldsymbol{u}^*(0)}(kP), \bar{\boldsymbol{u}}_{\boldsymbol{u}^*(0)}(kP)) = \qquad (3.15) \\ &\qquad \sum_{i=k}^{N/P-1} \tilde{L}(\tilde{x}_{\text{cMPC},\boldsymbol{u}^*(0)}(iP), \tilde{\bar{u}}^*(iP)) \\ &\qquad + \tilde{V}_f(\tilde{x}_{\text{cMPC},\boldsymbol{u}^*(0)}(kP+N)) \qquad (3.16) \end{aligned}$$

which is the rotated cost function with a candidate input sequence and

$$\begin{aligned} &J_{\text{aux}}(x_{\text{cMPC},\boldsymbol{u}^*(0)}(kP+P), \bar{\boldsymbol{u}}_{\boldsymbol{u}^*(0)}(kP+P)) = \\ &\qquad \sum_{i=k+1}^{N/P-1} \tilde{L}(\tilde{x}_{\text{cMPC},\boldsymbol{u}^*(0)}(iP), \tilde{\bar{u}}^*(iP)) \\ &\qquad + \tilde{L}(\tilde{x}_{\text{cMPC},\boldsymbol{u}^*(0)}(kP+N), \tilde{\kappa}_f(\tilde{x}_{\text{cMPC},\boldsymbol{u}^*(0)}(kP+N)) \\ &\qquad + \tilde{V}_f(\tilde{x}_{\text{cMPC},\boldsymbol{u}^*(0)}(kP+N+P)) \qquad (3.17) \end{aligned}$$

is the rotated cost of the (suboptimal) successor step using the candidate input trajectory (3.4). Using (3.16) and (3.17) we get

$$
\begin{aligned}
&J_{\text{aux}}(x_{\text{cMPC},\boldsymbol{u}^*(0)}(kP+P), \bar{\boldsymbol{u}}_{\boldsymbol{u}^*(0)}(kP+P)) \\
&\qquad\qquad - J_{\text{aux}}(x_{\text{cMPC},\boldsymbol{u}^*(0)}(kP), \bar{\boldsymbol{u}}_{\boldsymbol{u}^*(0)}(kP)) \\
&= -\tilde{L}(\tilde{x}_{\text{cMPC},\boldsymbol{u}^*(0)}(kP), \tilde{\bar{u}}^*(kP)) \\
&\quad + \tilde{L}(\tilde{x}_{\text{cMPC},\boldsymbol{u}^*(0)}(kP+N), \tilde{\kappa}_f(\tilde{x}_{\text{cMPC},\boldsymbol{u}^*(0)}(kP+N)) \\
&\quad - \tilde{V}_f(\tilde{x}_{\text{cMPC},\boldsymbol{u}^*(0)}(kP+N)) \\
&\quad + \tilde{V}_f(\tilde{x}_{\text{cMPC},\boldsymbol{u}^*(0)}(kP+N+P)) \\
&\underbrace{\leq}_{\text{Lem. 3.5.5}} -\tilde{L}(\tilde{x}_{\text{cMPC},\boldsymbol{u}^*(0)}(kP), \tilde{\bar{u}}^*(kP)) \\
&\underbrace{\leq}_{\text{Ass. 3.5.1}} -\alpha\left(|(\tilde{x}_{\text{cMPC},\boldsymbol{u}^*(0)}(kP), \tilde{\bar{u}}^*(kP))|_{\Pi}\right).
\end{aligned}
$$

□

Lemma 3.5.7. *The sequence* $J_{\text{aux}}(x_{\text{cMPC},\boldsymbol{u}^*(0)}(kP), \bar{\boldsymbol{u}}_{\boldsymbol{u}^*(0)}(kP))$ *converges for* $k \in \mathcal{I}_{\geq 0}, k \to \infty$. ■

Proof. By Lem. 3.5.6 we have

$$
\begin{aligned}
&J_{\text{aux}}(x_{\text{cMPC},\boldsymbol{u}^*(0)}(kP+P), \bar{\boldsymbol{u}}_{\boldsymbol{u}^*(0)}(kP+P)) \\
&\qquad\qquad - J_{\text{aux}}(x_{\text{cMPC},\boldsymbol{u}^*(0)}(kP), \bar{\boldsymbol{u}}_{\boldsymbol{u}^*(0)}(kP+P)) \\
&\leq -\alpha\left(|(\tilde{x}_{\text{cMPC},\boldsymbol{u}^*(0)}(kP), \tilde{\bar{u}}^*(kP))|_{\Pi}\right) \\
&\leq 0.
\end{aligned}
$$

Therefore the sequence $J_{\text{aux}}(x_{\text{cMPC},\boldsymbol{u}^*(0)}(kP), \bar{\boldsymbol{u}}_{\boldsymbol{u}^*(0)}(kP))$ is non-increasing with k. It is bounded from below, since J_{aux} is continuous and X_N is compact. It follows that the sequence must converge. □

Lemma 3.5.8. *For all* $x(0) \in \Pi_{\mathbb{X}}$ *it holds that* $J_{\text{aux}}(x_{\text{cMPC},\tilde{x}(0)}(kP), \bar{\boldsymbol{u}}_{\boldsymbol{u}^*(0)}(kP))$ *is constant for all* $k \in \mathcal{I}_{\geq 0}$. ■

Proof. By Ass. 3.5.2 it follows that $\bar{\boldsymbol{u}}_{\boldsymbol{u}^*(0)}(0) \in \Pi_{\mathbb{U}}^{N/P}$ and that the corresponding states follow the periodic orbit $\Pi_{\mathbb{X}}$. W.l.o.g. define

$$
\tilde{\pi}(\tilde{x}) := \begin{cases} (u_i^p, .., u_{P-1}^p, .., u_{i-1}^p), & \tilde{x} = (*, .., *, x_i^p) \in \tilde{\Pi}_{\mathbb{X}}, * \in \mathbb{X}, i \in \mathcal{I}_{[0,P-1]} \\ \tilde{\kappa}_f(\tilde{x}), & \text{else} \end{cases}
$$

as new, valid terminal controller w.r.t. Ass. 3.2.1. By using $\tilde{\pi}(\tilde{x})$ for constructing $\bar{\boldsymbol{u}}_{\boldsymbol{u}^*(0)}(kP)$ we always have a periodic completion according to Π in case $x(0) \in \Pi_{\mathbb{X}}$, i.e. $\bar{\boldsymbol{u}}_{\boldsymbol{u}^*(0)}(kP) \in \Pi_{\mathbb{U}}^{N/P}$ for all $k \in \mathcal{I}_{\geq 0}$. We conclude, that for all $k \in \mathcal{I}_{\geq 0}$ the arguments of J_{aux} are constant and therefore $J_{\text{aux}}(x_{\text{cMPC},\tilde{x}(0)}(kP+P), \bar{\boldsymbol{u}}_{\boldsymbol{u}^*(0)}(kP+P)) = J_{\text{aux}}(x_{\text{cMPC},\tilde{x}(0)}(kP), \bar{\boldsymbol{u}}_{\boldsymbol{u}^*(0)}(kP))$ which proves the statement. □

Lemma 3.5.9. *For all $\tilde{x}(t) \in \tilde{X}_N$ the function $V(.)$ (3.13) is finite.* ■

Proof. By noting the telescoping sum in $V(.)$ (3.13) we get

$$\begin{aligned} V(\tilde{x}) &= -\sum_{k=0}^{\infty}[J_{\text{aux}}(x_{\text{cMPC},\boldsymbol{u}^*(0)}(kP+P),\bar{\boldsymbol{u}}_{\boldsymbol{u}^*(0)}(kP+P)) \\ &\qquad\qquad - J_{\text{aux}}(x_{\text{cMPC},\boldsymbol{u}^*(0)}(kP),\bar{\boldsymbol{u}}_{\boldsymbol{u}^*(0)}(kP))] \\ &= -\lim_{k\to\infty} J_{\text{aux}}(x_{\text{cMPC},\boldsymbol{u}^*(0)}(kP+P),\bar{\boldsymbol{u}}_{\boldsymbol{u}^*(0)}(kP+P)) \\ &\qquad\qquad + J_{\text{aux}}(x_{\text{cMPC},\boldsymbol{u}^*(0)}(0),\bar{\boldsymbol{u}}_{\boldsymbol{u}^*(0)}(0)) \\ &= J_{\text{aux}}(x_{\text{cMPC},\boldsymbol{u}^*(0)}(0),\bar{\boldsymbol{u}}_{\boldsymbol{u}}) + \text{constant} \end{aligned}$$

using Lem. 3.5.7. J_{aux} is continuous in x and $\boldsymbol{u}$ because of the stage cost ℓ is assumed to be continuous in x and u, and V_f is continuous. Therefore, the functional J_{aux} is bounded since X_N is compact which completes the proof. □

Lemma 3.5.10. *The function $V(.)$ (3.13) is positive definite w.r.t. $\tilde{\Pi}_{\mathbb{X}}$.* ■

Proof. Using Lem. 3.5.6 we have

$$\begin{aligned} V(\tilde{x}) &= -\sum_{k=0}^{\infty}[J_{\text{aux}}(x_{\text{cMPC},\boldsymbol{u}^*(0)}(kP+P),\bar{\boldsymbol{u}}_{\boldsymbol{u}^*(0)}(kP+P)) \\ &\qquad\qquad - J_{\text{aux}}(x_{\text{cMPC},\boldsymbol{u}^*(0)}(kP),\bar{\boldsymbol{u}}_{\boldsymbol{u}^*(0)}(kP))] \\ &\geq \sum_{k=0}^{\infty} \alpha\left(|(\tilde{x}_{\text{cMPC},\boldsymbol{u}^*(0)}(kP), \tilde{\bar{u}}^*(kP))|_{\Pi}\right) \\ &\geq \sum_{k=0}^{\infty} \alpha\left(|\tilde{x}_{\text{cMPC},\boldsymbol{u}^*(0)}(kP)|_{\Pi_{\mathbb{X}}}\right). \end{aligned}$$

This implies that $V(\tilde{x}) > 0$ for all $\tilde{x} \notin \tilde{\Pi}_{\mathbb{X}}$. If $\tilde{x} \in \tilde{\Pi}_{\mathbb{X}}$ we have by Lem. 3.5.8 that $V(\tilde{x}) = 0$ and by Lem. 3.5.9 that V is bounded from above. □

Lemma 3.5.11. *There exists a $\mathcal{K}_\infty$ function $\alpha_1(.)$ such that for all $\tilde{x} \in X_N$ we have $V(\tilde{x}) \geq \alpha_1(|\tilde{x}|_{\Pi_\mathbb{X}})$.* ■

Proof. Lem. 3.5.10 states that $V(\tilde{x})$ is positive definite w.r.t. $\Pi_\mathbb{X}$. Analogously to [13, p. 341] we define the non-decreasing function

$$\hat{\alpha}(|\tilde{x}|_{\Pi_\mathbb{X}}) := \min_{\bar{x} \in \tilde{X}_N : |\bar{x}|_{\Pi_\mathbb{X}} \geq |\tilde{x}|_{\Pi_\mathbb{X}}} V(\bar{x}).$$

Using the lower bound in the proof of Lem. 3.5.10 one can show that $\hat{\alpha}(|\tilde{x}|_{\Pi_\mathbb{X}}) \to \infty$ as $|\tilde{x}|_{\Pi_\mathbb{X}} \to \infty$. Therefore $\hat{\alpha}$ is radially unbounded. Since $\hat{\alpha}(|\tilde{x}|_{\Pi_\mathbb{X}})$ is strictly positive for $|\tilde{x}|_{\Pi_\mathbb{X}} > 0$, it is possible to lower bound this non-decreasing function by one that is strictly increasing and radially unbounded [13, p. 341] which we choose to be α_1 to obtain the desired result. □

Lemma 3.5.12. *There exist a $\mathcal{K}_\infty$ function $\alpha_2(|\tilde{x}|_{\Pi_\mathbb{X}})$ such that $V(\tilde{x}) \leq \alpha_2(|\tilde{x}|_{\Pi_\mathbb{X}})$.* ■

Proof. Define

$$\alpha_2(|\tilde{x}|_{\Pi_\mathbb{X}}) := \max_{\bar{x} \in \tilde{X}_N : |\bar{x}|_{\Pi_\mathbb{X}} \leq |\tilde{x}|_{\Pi_\mathbb{X}}} V(\bar{x}) + |\tilde{x}|_{\Pi_\mathbb{X}}.$$

The first summand is non-decreasing, with increasing $|\tilde{x}|_{\Pi_\mathbb{X}}$ and finite by Lem. 3.5.9. By optimality it is greater or equal than $V(\tilde{x})$. The second summand is strictly increasing in $|\tilde{x}|_{\Pi_\mathbb{X}}$. Therefore $\alpha_2(|\tilde{x}|_{\Pi_\mathbb{X}})$ is strictly increasing. Further, the second summand is radially unbounded which completes the proof. □

Theorem 3.5.13 (Asymptotic stability)**.** *If Ass. 3.2.1, 3.2.3, 3.5.1 and 3.5.2 are fulfilled, then under application of Alg. 1 the P-step system according to the system (1.5) is asymptotic stable w.r.t. the set $\tilde{\Pi}_\mathbb{X}$ with region of attraction[3] $\tilde{X}_N$.* ■

Proof. From Lem. 3.5.11 and Lem. 3.5.12 there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$V(\tilde{x}(t)) \geq \alpha_1(|\tilde{x}(t)|_{\Pi_\mathbb{X}})$$
$$V(\tilde{x}(t)) \leq \alpha_2(|\tilde{x}(t)|_{\Pi_\mathbb{X}}).$$

[3] The region of attraction is the set from which we can reach the terminal region with respect to the last state of the P-step system state.

Further by the definition of $V(.)$ (3.13) and Lem. 3.5.6, we have for $t \in \mathcal{I}_{\geq 0}$

$$\begin{aligned}
V(\tilde{x}_{\mathrm{cMPC},\boldsymbol{u}^*(0)}(t+P)) - V(\tilde{x}_{\mathrm{cMPC},\boldsymbol{u}^*(0)}(t))& \\
&= J_{\mathrm{aux}}(x_{\mathrm{cMPC},\boldsymbol{u}^*(0)}(t+P), \bar{\boldsymbol{u}}_{\boldsymbol{u}^*(0)}(t+P)) \\
&\quad - J_{\mathrm{aux}}(x_{\mathrm{cMPC},\boldsymbol{u}^*(0)}(t), \bar{\boldsymbol{u}}_{\boldsymbol{u}^*(0)}(t)) \\
&\leq -\alpha(|(\tilde{x}_{\mathrm{cMPC},\boldsymbol{u}^*(0)}(t), \tilde{u}^*(t))|_{\Pi}) \\
&\leq -\alpha(|\tilde{x}_{\mathrm{cMPC},\boldsymbol{u}^*(0)}(t)|_{\Pi_{\mathbb{X}}}).
\end{aligned}$$

Since $\alpha \in \mathcal{K}_{\infty}$ it follows by e.g. [20, Thm. B.13] asymptotic stability for the closed loop P-step system using the candidate input sequence. By optimality we have

$$\begin{aligned}
J_{\mathrm{aux}}(\tilde{x}_{\mathrm{cMPC},\boldsymbol{u}^*(0)}(t+P), \boldsymbol{u}^*(t+P))& \\
&\leq J_{\mathrm{aux}}(\tilde{x}_{\mathrm{cMPC},\boldsymbol{u}^*(0)}(t+P), \bar{\boldsymbol{u}}_{\boldsymbol{u}^*(0)}(t+P))
\end{aligned}$$

from which it follows also asymptotic stability of the closed loop system under application of Alg. 1. □

Remark 3.5.14. *Asymptotic stability of the P-step system according to the nominal system (4.1) w.r.t. the set of shifted trajectories regarding the orbit Π means, that the nominal system periodically converges to the orbit Π. In particular this does not mean, that the nominal system asymptotically converges to $\Pi_{\mathbb{X}}$, since it could be that we move away from the set $\tilde{\Pi}_{\mathbb{X}}$ at some time instance during P consecutive steps. Further, in case $\Pi_{\mathbb{X}}$ contains a steady-state $\bar{x} \in \Pi_{\mathbb{X}}$ we can guarantee that we do not converge to the steady state, as it could be the case in e.g. [24], but to periodic operation.* ■

Remark 3.5.15. *The stability proof presented in this Section is more general than the classical proof provided in [1], because it can be used to prove stability w.r.t. to arbitrary sets $\mathcal{A}$ in case of strict dissipativity, i.e. using $\alpha(|\tilde{x}|_{\mathcal{A}})$.* ■

3.6 Related work: Economic MPC without terminal constraints

For comparison we introduce a related method [18] and give the respective theoretical results (asymptotic average performance and stability w.r.t. to the optimal periodic orbit). The main difference is, that the method does not need a terminal set and terminal cost. Consider the open loop optimization problem

$$(P_{\text{EMPCU-P}}) \begin{cases} \min_{\boldsymbol{u}\in\mathbb{U}^N} \sum_{k=0}^{N-1} \ell(x_{\boldsymbol{u}}(k,x),u(k)) \\ \text{s.t. for all } k \in \mathcal{I}_{[t,t+N-1]} : \\ \qquad x_{\boldsymbol{u}}(k+1,x) = f(x_{\boldsymbol{u}}(k),u(k),0) \\ \qquad x_{\boldsymbol{u}}(k,x) \in \mathbb{X} \\ \qquad u(k) \in \mathbb{U} \\ \qquad x_{\boldsymbol{u}}(0,x) = x. \end{cases}$$

The MPC scheme [18] is given in Alg. 2 for easier reference.

Assumptions

In addition to the strict dissipativity assumption (Ass. 3.5.1) there are additional controllability assumptions, required in [18].

Assumption 3.6.1 (Local controllability on $\mathcal{B}_\kappa(\Pi_\mathbb{X})$ [18])**.** *There exists $\kappa > 0$, $M' \in \mathcal{I}_{\geq 0}$ and $\rho \in \mathcal{K}_\infty$ such that for all $z \in \Pi_\mathbb{X}$ and all $x,y \in \mathcal{B}_\kappa(z) \cap \mathbb{X}$ there exists a control sequence $\boldsymbol{u} \in \mathbb{U}^{M'}(x)$ such that $x_{\boldsymbol{u}}(M',x) = y$ and*

$$|(x_u(k,x),u(k))|_\Pi \leq \rho(\max\{|x|_{\Pi_\mathbb{X}},|y|_{\Pi_\mathbb{X}}\})$$

holds for all $k \in \mathcal{I}_{[0,M'-1]}$. ■

Assumption 3.6.2 (Finite Time Controllability into $\mathcal{B}_\kappa(\Pi_\mathbb{X})$ [18])**.** *For $\kappa > 0$ from Ass. 3.6.1 there exists $M'' \in \mathcal{I}_{\geq 0}$ sucht that for each $x \in \mathbb{X}$ there exists $k \in \mathcal{I}_{[0,M'']}$ and $u \in \mathbb{U}^k(x)$ such that $x_u(k,x) \in \mathcal{B}_\kappa(\Pi_\mathbb{X})$.* ■

Assumption 3.6.3 (Global recursive feasibility)**.** *For all $x \in \mathbb{X}$ it holds $\mathbb{U}^\infty(x) \neq \emptyset$.* ■

Algorithm 2 Economic MPC without terminal constraints for optimal periodic operation

---

1: **procedure** EMPCU-P(initial state $x(0)$)
2:     **for** $k = 0,1,\ldots$ **do**
3:         solve $(P_{\text{EMPCU-P}})$ with initial condition $x_{\text{MPC}}(kP)$
4:         apply the first P inputs of $\boldsymbol{u}^*(kP)$ to the system (1.5)
5:     **end for**
6: **end procedure**

Remark 3.6.4. *Ass. 3.6.3 ensures, that Alg. 2 is recursive feasibility. This is a rather restrictive assumption and a general drawback of MPC schemes without a terminal set with a terminal controller. Because it is not possible to use the terminal controller in order to construct the feasible input sequence as in Thm. 3.3.1.* ■

Asymptotic average performance

If the controllability and recursive feasibility assumptions hold and in case of strict dissipativity, it follows that the closed loop asymptotic average performance under application of Alg. 2 is nearly optimal. Since there is no terminal cost, the same closed loop performance can only be achieved in case $N \to \infty$.

Corollary 3.6.5 (Asymptotic average performance [18])**.** *Consider the P-step MPC scheme as defined via Alg. 2 and suppose that Ass. 3.5.1, 3.6.1, 3.6.2 and 3.6.3 are satisfied for some minimal P-periodic orbit $\Pi \subseteq \mathrm{int}(\mathbb{X} \times \mathbb{U})$ of system (1.5) and with $M' = iP$ for some $i \in \mathcal{I}_{\geq 1}$. Furthermore, assume that f and ℓ are continuous and ℓ is bounded on $\mathbb{X} \times \mathbb{U}$. Then system (1.5) is optimally operated at the periodic orbit Π and there exist $\delta_1, \delta_2 \in \mathcal{L}$ and $\bar{N} \in \mathcal{I}_{\geq 1}$ such that for the resulting closed-loop system, the performance estimate*

$$\limsup_{T \to \infty} \frac{\sum_{k=t}^{T-1} \ell(x_{\mathrm{MPC}}(k), u_{\mathrm{MPC}}(k))}{T} \leq \tag{3.18}$$

$$(1/P) \sum_{k=0}^{P-1} \ell(x_k^p, u_k^p) + \delta_1(N-P)/P + \delta_2(N-P) \tag{3.19}$$

is satisfied for all $x \in \mathbb{X}$, all $N \in \mathcal{I}_{\geq \bar{N}+P}$ and all $K \in \mathcal{I}_{\geq 0}$. ■

Proof. The verification can be found in [18]. □

Practical asymptotic convergence to optimal orbit

Using the same conditions as needed for nearly optimal performance in the previous section, in [18] it is also shown, that the closed loop under Alg. 2 will converge into a neighbourhood of the optimal periodic orbit Π. Similarly, the neighbourhood can be made arbitrarily small by choosing N large enough.

Theorem 3.6.6 (Practical asymptotic convergence to optimal orbit [18])**.** *Suppose that the conditions of Cor. 3.6.5 are satisfied and that the storage function $\tilde{\lambda}$ is continuous on $\mathbb{X}^P$. Then there exists $\hat{N} \in \mathcal{I}_{\geq 1}$ such that for all*

$N \in \mathcal{I}_{\geq \hat{N}}$, the closed-loop system resulting from application of the P-step MPC scheme defined in Alg. 2 practically asymptotically converges to the optimal periodic orbit Π, i.e., there exists $\nu \in \mathcal{L}$ and for each $x \in \mathbb{X}$ some $\hat{k} \in \mathcal{I}_{\geq 0}$ and $j \in \mathcal{I}_{[0,P-1]}$ such that $(x_{\mathrm{MPC}}(k), u_{\mathrm{MPC}}(k)) \in \mathcal{B}_{\nu(N)}\left(x^p_{[k+j]}, u^p_{[k+j]}\right)$ for all $k \in \mathcal{I}_{\geq \hat{k}}$. ■

Proof. The proof can be found in [18]. □

3.7 Example: Simple supply chain network

Consider the simple supply chain example, introduced in Sec. 1.3. For easier reference, consider the states

$$x := [x_{S,1}, x_{S,2}, x_{T,P}, x_{T,L}, x_R]^\top.$$

Simple supply chain: Economic MPC with terminal cost and constraint

In order to apply Alg. 1, we have to make sure that all assumptions are satisfied. In Sec. 2.5 we already discussed strict dissipativity. For Ass. 3.2.1 choose the terminal set

$$\mathbb{X}_f := \left\{ x \in \{\mathbb{X} \cap \{ \left\{ \begin{array}{c} x_{S,1} = 0 \\ x_{S,2} = 0 \\ x_{T,P} = 1 \\ x_{T,L} = 2 - x_R \\ x_R \leq 0 \end{array} \right\} \cup \left\{ \begin{array}{c} x_{S,1} = 2 \\ x_{S,2} = 0 \\ x_{T,P} = 0 \\ x_{T,L} = 0 \\ x_R = 1 \end{array} \right\} \}\} \right\},$$

terminal controller

$$\tilde{\kappa}_f(\tilde{x}) := \begin{cases} \left(\begin{bmatrix} 1 \\ x_R - 2 \\ 2 - x_R - x_{T,L} + x_{S,2} \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right), & x_{T,P} = 1 \\ ([1,2,0]^\top, [1, -2, 2]^\top), & x_{T,P} = 0 \end{cases}$$

with $\tilde{x} = ((*), x)$, $(*) \in \mathbb{R}^5$ and terminal cost defined as

$$V_f(x) := x_{T,P}(-11 x_R).$$

We can interpret the terminal setting as follows: Consider the subset of $\mathbb{X}_f$ in which we fix the possible graph configuration such that the truck is located at the retailer. The constraints are chosen such that the retail store can have optimal ($= 0$) or negative (suboptimal) values which increases the initial feasible region of ($P_{\text{EMPC-P}}$). The truck however must contain as many goods as needed in order to fill up the retail store to the optimal value ($= 0$) plus the value of the optimal orbit. This is done by the first three inputs of the first part of the terminal controller. The second three inputs of the first part of $\tilde{\kappa}_f$ then equal the optimal periodic orbit. In case, the the truck is located at the supplier (second case of $\mathbb{X}_f$) we simply use the optimal input sequence (second part of $\tilde{\kappa}_f$). We verify the terminal setting by minimizing Ass. 3.2.1, 3. Since the minimization problem is convex for a given truck position and we obtained 0 as a result for every truck position it follows that the terminal set, controller and cost fulfill Ass. 3.2.1. In addition, by solving ($P_{\text{EMPC-P}}$) for every $x \in \Pi_{\mathbb{X}}$, we obtained the corresponding periodic input trajectory, i.e. $\boldsymbol{u}^* \in \Pi_{\mathbb{U}}^{N/P}$ for various[4] $N = N_1 P$, $N_1 \in \mathcal{I}_{\geq 1}$ and therefore Ass. 3.5.2 is satisfied.

In Fig. 3.1 the closed loop results are shown using a prediction horizon $N = 2$, starting off the optimal periodic orbit. The bottom plot in Fig. 3.1 shows the distance to the optimal periodic orbit. As mentioned in Rem. 3.5.14 we can see, that we keep the distance to the optimal periodic orbit within one time instance at the $t = 1$. But we asymptotically converge to the optimal orbit in P-step system steps.

Simple supply chain: Economic MPC without terminal cost and constraint

In order to apply Alg. 2 we have to check Ass. 3.6.1 and Ass. 3.6.2. Again we already discussed strict dissipativity in Sec. 2.5. For verification of the controllability assumptions we used the Multi-Parametric-Toolbox [12]. Starting at the optimal periodic orbit, we calculated the forward and backward reachable sets. This can be done easily by considering the P-step system representation and fixing the truck trajectory to the optimal periodic orbit. This yields a linear time invariant system representation. It turns out that the forward and backward invariant sets converge to the feasible set (1.16), from which we conclude that Ass. 3.6.2 and Ass. 3.6.2 hold. Note, that since $\Pi \subseteq \text{int}(\mathbb{X} \times \mathbb{U})$ does not hold in our example, Cor. 3.6.5 holds neither.

[4]All N that will be used in the experiments.

In Fig. 3.2 we show the closed loop distance to the optimal periodic orbit for different planning horizons using the same initial condition as in the previous paragraph. As expected, the longer the planning horizon, the faster we converge to the neighbourhood of the optimal orbit. For the particular initial condition considered here, we must choose a planning horizon $N \geq 6$ in order to converge exactly to the optimal orbit. In Fig. 3.3 a sample closed loop trajectory for $N = 4$ is shown.

Comparison: With disturbances

We compare both Algorithms in Tab. 3.2 under the presence of uniformly distributed disturbances

$$w(k) \in \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -3 \end{bmatrix} \right\}. \tag{3.20}$$

Therefore we took the average performance for different planning horizons by simulating 4000 time steps. Alg. 1 yields for $N = 4$ similar performance as Alg. 2 for $N = 5$. The terminal constrained version leads to infeasible online optimization problems for $N = 2$ due to the disturbances. The unconstraint version is unstable/infeasible for $N = 3$ and does not yield any better performance for planning horizons $N > 5$. This is surprising, because as shown in Fig. 3.2, for a special initial condition we need $N = 6$ to actually converge to the optimal orbit. In Fig. 3.4 a sample closed loop trajectory of Alg. 1 under disturbances is shown, and in Fig. 3.5 for Alg. 2 respectively.

Table 3.2: Comparison of Alg. 1 and Alg. 2 applied to the simple supply chain model under disturbances and with different planning horizons. Shown is the approximate average performance for 4000 time steps.

Algorithm	Planning hor. N	Avg. Performance
Alg. 1	2	infeasible
Alg. 1	4	36.71
Alg. 2	3	$\to \infty$
Alg. 2	4	45.92
Alg. 2	5	35.29
Alg. 2	6	36.07

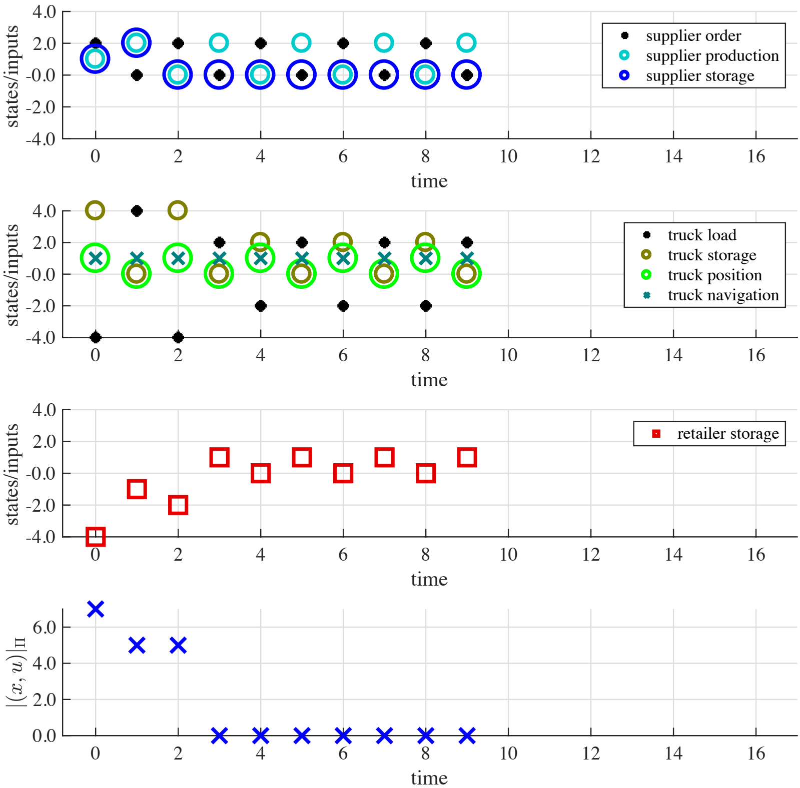

Figure 3.1: Closed loop results of the simple supply chain network using Alg. 1 and $N = 2$.

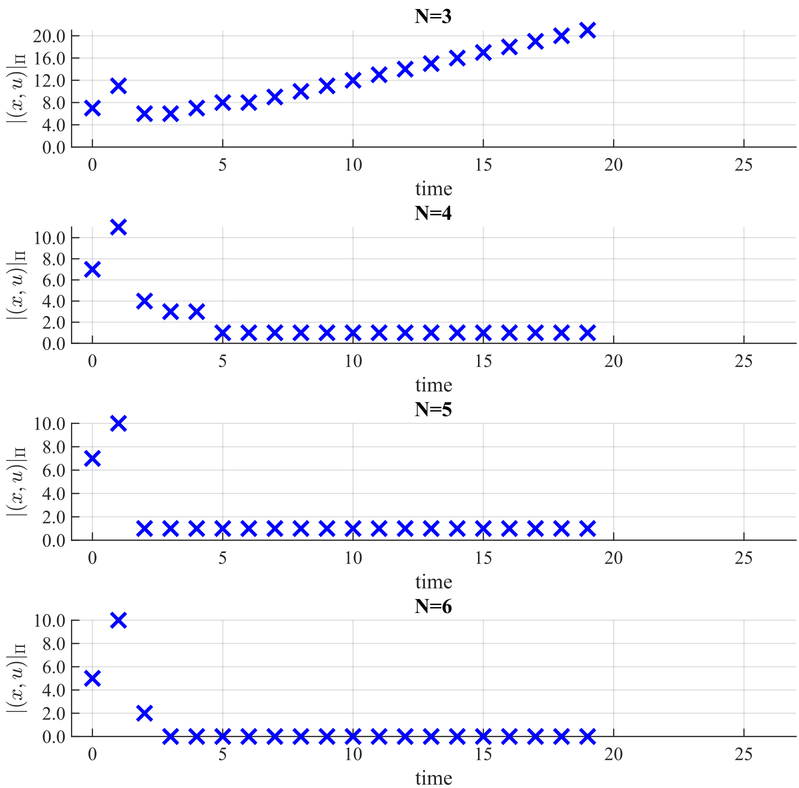

Figure 3.2: Distance to the optimal periodic orbit of the closed loop system under application of Alg. 2 with different planning horizons.

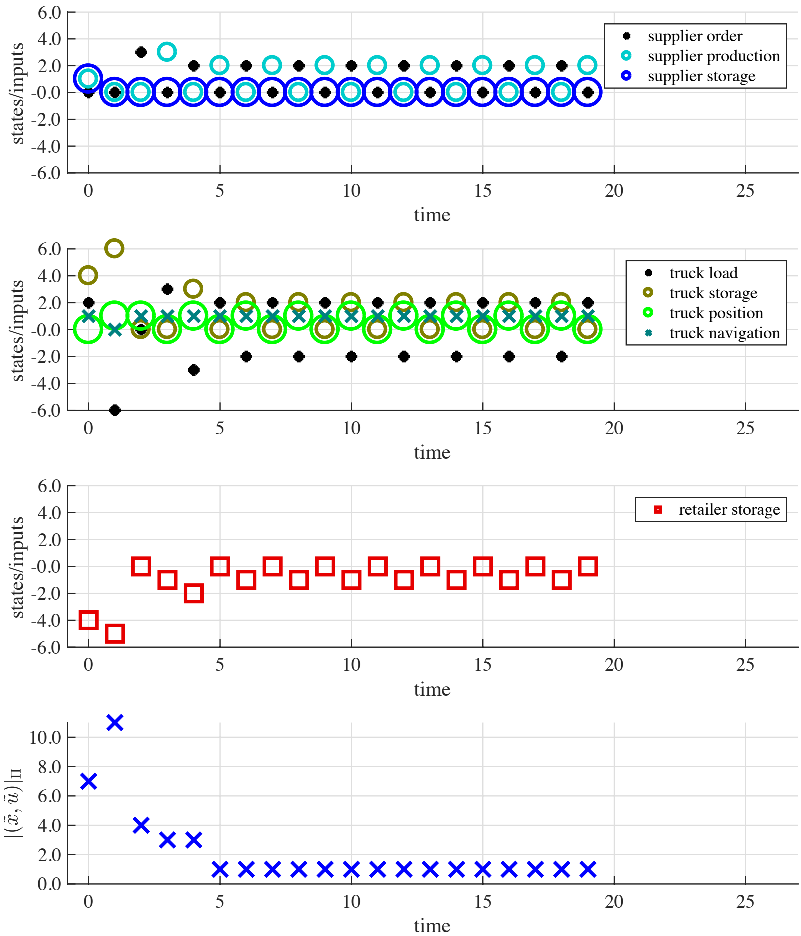

Figure 3.3: Closed loop results of the simple supply chain network using Alg. 2 and planning horizon $N = 4$.

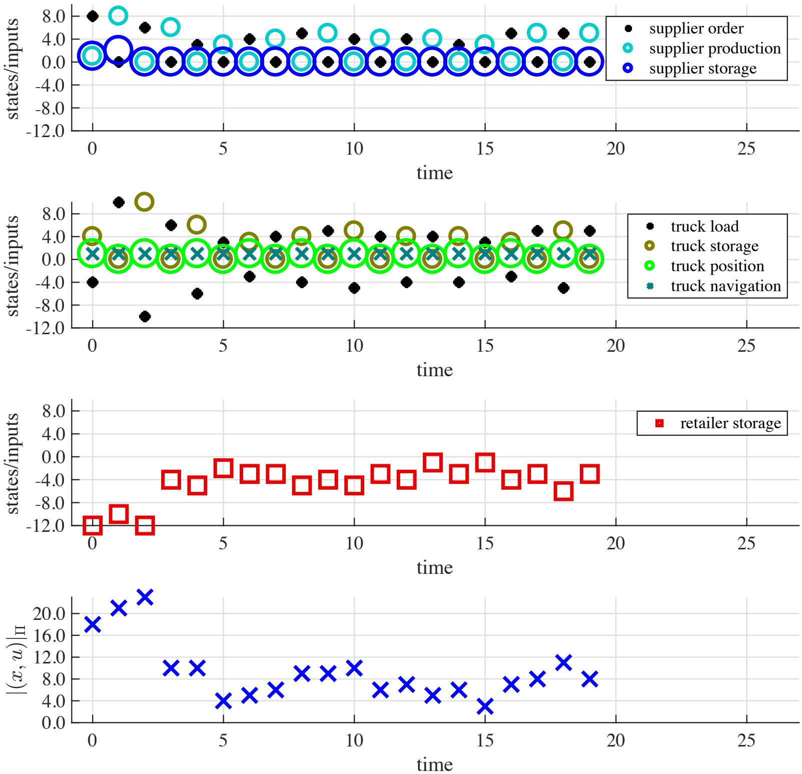

Figure 3.4: Closed loop results of the simple supply chain network *under disturbances* using Alg. 1 and planning horizon $N = 4$.

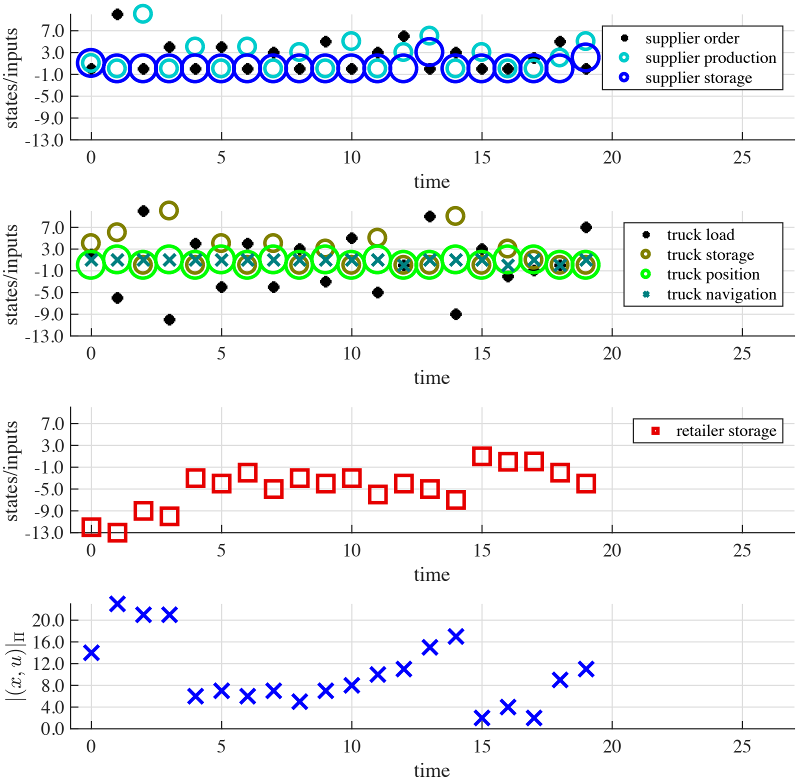

Figure 3.5: Closed loop results of the simple supply chain network *under disturbances* using Alg. 2 and planning horizon $N = 6$.

4 Tube-based robust economic MPC for periodic operation

4.1 Introduction

In applications, external disturbances can have significant impact to the system dynamics. In most cases this leads to poor performance as demonstrated in the previous chapter, or even in a loss of feasibility which could lead to safety risks. In order to overcome the afore mentioned drawbacks, in [4] a robust economic control scheme was developed. It is based on an auxiliary controller, which keeps the disturbed system state inside a tube around a nominal (virtual) system state. This way, knowledge about the disturbances which act on the system can be explicitly exploited in the process of controller design, yielding performance improvements and safety guarantees.

Using the concept of [4], we study the case in which *periodic* operation is optimal, instead of steady state operation. We develop a tube-based robust economic model predictive control scheme for robust optimal periodic operation. To this end we extend the notion of robust optimal steady-state operation of a system to robust optimal periodic operation. The algorithm we propose is proven to be recursively feasible and has an average performance which is no worse than that of robust optimal periodic operation. By linking recent results regarding a certain dissipativity inequality as sufficient condition for optimal periodic operation [11, 19] to the case of robust optimal periodic operation, we establish a checkable sufficient condition that implies that a system is robustly optimally operated at periodic operation.

Finally, if a stronger dissipativity criterion, namely strict dissipativity is fulfilled, our control scheme is then proven to asymptotically stabilize the 'tube' around the robust optimal periodic orbit.

4.2 Invariant error sets

Definition 4.2.1 (Error dynamics). Define the nominal system as

$$z(t+1) = f(z(t),v(t),0),\ z(0) = z. \tag{4.1}$$

Let the error between the real, disturbed system state $x(t)$ and the nominal system state $z(t)$ be defined as $e(t) := x(t) - z(t)$. Further let

$$u(t) = \phi(v(t),x(t),z(t)) \in \mathbb{U} \tag{4.2}$$

be an error feedback in order to keep the real system state $x(t)$ close to the nominal system state $z(t)$. The error dynamics is then defined as

$$e(t+1) = f(x(t),\phi(v(t),x(t),z(t)),w(t)) - f(z(t),v(t),0). \tag{4.3}$$

■

Definition 4.2.2 (Robustly control invariant set [4]). A set $\Omega \subseteq \mathbb{R}^n$ is robustly control invariant (RCI) for the error system (4.3) if and only if there exists a feedback law of the form (4.2) such that for all $x(t),z(t) \in \mathbb{X}$ with $e(t) \in \Omega$ and all $v(t) \in \mathbb{U}, w \in \mathbb{W}$ it holds that $e(t+1) \in \Omega$ and $u(t) \in \mathbb{U}$. ■

The concept of tube-based robust model predictive control is to perform the open-loop optimization for the nominal system and then apply the input according to (4.2) to the real system. This way we guarantee that the real, disturbed system state $x(t)$ will always stay within an compact RCI set Ω around the nominal, calculated (predicted) states $z(t)$.

Remark 4.2.3 (Tightened constraints for nominal system [4]). *In order to guarantee that $(x,u) \in \mathbb{X} \times \mathbb{U}$ under application of (4.2) we must tighten the state and input constraints $\mathbb{X}$ and $\mathbb{U}$ of (4.1) to*

$$\bar{\mathbb{Z}} := \{(z,v) \in \mathbb{X} \times \mathbb{U} | (x,\phi(v,x,z)) \in \mathbb{X} \times \mathbb{U} \textit{ for all } x \in \{z\} \oplus \Omega\}. \tag{4.4}$$

In the following we denote the projection of $\bar{\mathbb{Z}}$ on $\mathbb{X}$ as $\bar{\mathbb{X}}$ and the projection on $\mathbb{U}$ as $\bar{\mathbb{U}}$ respectively. ■

4.3 Robust periodic cost functional

Assumption 4.3.1 (Existence of RCI set). *There exists an $\phi : \bar{\mathbb{U}} \times \mathbb{X} \times \bar{\mathbb{X}} \to \mathbb{U}$ according to (4.2) providing an appropriate control law, that implies an RCI set Ω for error dynamics (4.3).* ■

Definition 4.3.2 (Robust optimal periodic orbit)**.** Let Ass. 4.3.1 hold. The robust optimal periodic orbit Π^* with optimal period length P^* of a disturbance-affected system (1.5) for a given stage cost function (1.6) is defined as

$$\{P^*, \Pi^*\} = \underset{P \in \mathcal{I}_{\geq 1}, \Pi \in S_\Pi^P}{\operatorname{argmin}} \frac{1}{P} \sum_{i=0}^{P-1} \left(\int_{x_i^p \in \{z_i^p\} \oplus \Omega} \ell(x_i^p, \phi(v_i^p, x_i^p, z_i^p)) dx \right) \quad (4.5)$$

with $(z_i^p, v_i^p) \in \Pi^*$ and with Ω an RCI set. In case of multiple solution pairs choose any solution with minimal period length. ■

Remark 4.3.3. *Since the real, disturbance-affected system stays within the tube around the robust optimal periodic orbit, Def. 4.3.2 defines the best periodic orbit for the nominal system, when averaging the cost over all possible disturbance affected trajectories of the real system inside the tube. The definition could be more precise by incorporating stochastic information regarding the disturbances.* ■

Next, we define the integrated stage cost function for system (1.5) and the corresponding integrated stage cost for the P-step system (1.9).

Definition 4.3.4 (Integrated stage cost function [4])**.** Consider system (1.5) with stage cost function (1.6). Given an error feedback $u = \phi(v,x,z)$ and a corresponding RCI set Ω, the integrated stage cost function is defined as

$$\ell^{\text{int}}(z,v) := \int_{x \in \{z\} \oplus \Omega} \ell(x, \phi(v,x,z)) dx. \quad (4.6)$$

■

Definition 4.3.5 (Integrated stage cost function of P-step system)**.** Consider the P-step system (1.9) according to system (1.5). Given an error feedback $u = \phi(v,x,z)$ and a corresponding RCI set Ω, the integrated P-step system stage cost function is defined as

$$\tilde{\ell}^{\text{int}}(\tilde{z},\tilde{v}) := \sum_{i=0}^{P-1} \int_{\tilde{x}_i \in \{\tilde{z}_{\tilde{v}}(i, z_{P-1})\} \oplus \Omega} \ell\left(\tilde{x}_i, \phi(\tilde{v}_i, \tilde{x}_i, \tilde{z}_{\tilde{v}}(i, z_{P-1}))\right) dx. \quad (4.7)$$

■

4.4 Assumptions and algorithm

In the following we use the same notation as described in Tab. 3.1. The same notation is also used for the nominal states by replacing x with z and u with v respectively.

Assumption 4.4.1 (Terminal controller, set and cost). *Let $(z_i^p, v_i^p) \in \Pi$ for $i \in \mathcal{I}_{[0,P-1]}$ and let Ass. 4.3.1 hold. There exists a compact $\bar{\mathbb{X}}_f \subseteq \bar{\mathbb{X}}$ such that for all $i \in \mathcal{I}_{[0,P-1]}$ the set of phase shifted orbits $\tilde{\Pi}_{\mathbb{X}}$ of $\Pi_{\mathbb{X}}$ is contained in $\bar{\mathbb{X}}_f^P$. Further assume that there exists a feedback law $\tilde{\bar{\kappa}}_f : \bar{\mathbb{X}}^P \to \bar{\mathbb{U}}^P$ and a continuous terminal cost $\bar{V}_f : \bar{\mathbb{X}}_f \to \mathbb{R}$ such that $\forall \tilde{z}$ with $z_{P-1} \in \bar{\mathbb{X}}_f$:*

1. $\tilde{\bar{\kappa}}_f(\tilde{z}) \in \bar{\mathbb{U}}^P$ $\qquad \triangleright$ *feasibility*
2. $f^P(\tilde{z}, \tilde{\bar{\kappa}}_f(\tilde{z}), 0) \in \bar{\mathbb{X}}_f^P$ $\qquad \triangleright$ *positive invariance of* $\bar{\mathbb{X}}_f^P$
3. $$\bar{V}_f((f^P(\tilde{z}, \tilde{\bar{\kappa}}_f(\tilde{z}), 0))_{P-1}) - \bar{V}_f((\tilde{z})_{P-1}) \leq -\tilde{\ell}(\tilde{z}, \tilde{\bar{\kappa}}_f(\tilde{z})) + \sum_{i=0}^{P-1} \ell^{\mathrm{int}}(z_i^p, v_i^p).$$

Without loss of generality assume $V_f(z) \geq 0 \; \forall z \in \bar{\mathbb{X}}_f$. ■

Remark 4.4.2. *Ass. 4.4.1 is the modification of Ass. 3.2.1 w.r.t. to the tightened sets and the integrated stage cost function.* ■

Let $N = N_1 P$ with $N_1 \in \mathcal{I}_{>0}$. Consider

$$(P_{\text{REMPC-P}}) \begin{cases} \min_{\boldsymbol{v} \in \bar{\mathbb{U}}^N} J^{\mathrm{int}}_{\mathrm{MPC}}(z, \boldsymbol{v}) \\ \text{s.t. for all } k \in \mathcal{I}_{[0,N-1]} : \\ \qquad z_{\boldsymbol{v}}(k+1, z) = f(z_{\boldsymbol{v}}(k,z), v(k), 0) \\ \qquad (z_{\boldsymbol{v}}(k,z), v(k)) \in \bar{\mathbb{Z}} \\ \qquad z_{\boldsymbol{v}}(N,z) \in \bar{\mathbb{X}}_f \\ \qquad z_{\boldsymbol{v}(t)}(0,z) = z \end{cases}$$

with finite time open loop cost

$$\begin{aligned} J^{\mathrm{int}}_{\mathrm{MPC}}(z, \boldsymbol{v}) := & \sum_{k=0}^{N-1} \ell^{\mathrm{int}}(z_{\boldsymbol{u}}(k,z), v(k)) + \bar{V}_f(z_{\boldsymbol{v}}(N,z)) \\ = & \sum_{k=0}^{N/P-1} \tilde{\ell}^{\mathrm{int}}(\tilde{z}_{\boldsymbol{v}}(kP, z), \tilde{v}(kP)) + \bar{V}_f(z_{\boldsymbol{v}}(N, z)). \end{aligned} \tag{4.8}$$

Assumption 4.4.3. *The optimization problem* $(P_{\text{REMPC-P}})$ *is feasible at time* $t = 0$. ■

In Alg. 3 the P-step robust economic model predictive control algorithm for optimal periodic operation is shown. If Alg. 3 is applied to system (1.5) we denote the closed loop dynamics by

$$z_{\text{MPC}}(t+1) = f(z_{\text{MPC}}(t), v_{\text{MPC}}(t), 0) \tag{4.9}$$

of the nominal system and

$$x_{\text{MPC}}(t+1) = f(x_{\text{MPC}}(t), u_{\text{MPC}}(t), w(t)) \tag{4.10}$$

for the real system.

4.5 Recursive feasibility and asymptotic average performance

Theorem 4.5.1 (Recursive feasibility of $(P_{\text{REMPC-P}})$)**.** *If Ass. 4.3.1, Ass. 4.4.1 and Ass. 4.4.3 hold, then Alg. 3 is recursively feasible.* ■

Proof. The proof can be done analogously to the proof of Thm. 3.3.1 for recursive feasibility of the nominal system (4.9). In particular, by Ass. 4.3.1 it immediately follows recursive feasibility of the real system (4.10). □

Algorithm 3 Robust economic model predictive control for optimal periodic operation

procedure REMPC-P(initial state $x(0)$)
  for $k_1 = 0, P, 2P, ..$ **do**
    solve $(P_{\text{REMPC-P}})$
    for $k_2 = 0,1,..,P-1$ **do**
      $u_{\text{MPC}}(k_1+k_2) = \phi(v^*(k_1+k_2), x_{\text{MPC}}(k_1+k_2), z_{\text{MPC}}(k_1+k_2))$
      $v_{\text{MPC}}(k_1+k_2) = v^*(k_1+k_2)$
    end for
  end for
end procedure

Theorem 4.5.2. *If Ass. 4.4.1 and Ass. 4.4.3 hold, then under application of Alg. 3 the closed loop system (4.10) has a robust average performance which is no worse than that of the robust optimal periodic orbit $\{P^*, \Pi^*\}$, i.e.*

$$\frac{1}{P^*}\sum_{k=0}^{P^*-1} \ell^{\text{int}}(z_k^p, v_k^p) \geq \limsup_{T\to\infty} \frac{\sum_{t=0}^{T-1} \ell^{\text{int}}(z_{\text{MPC}}(t), v_{\text{MPC}}(t))}{T} \tag{4.11}$$

with $(z_k^p, v_k^p) \in \Pi^*$ for $k \in \mathcal{I}_{[0,P^*-1]}$. ■

Proof. The proof follows the lines of the proof of Thm. 3.4.1 by setting $\ell := \ell^{\text{int}}$ and $V_f := \bar{V}_f$. □

4.6 Robust optimal periodic operation

In this section we extend the concept of robust optimal steady-state operation, as it was introduced in [4], to the case of robust optimal periodic operation. Therefore we adapt [4, Def. 4] with respect to [18, Def. 3].

Definition 4.6.1 (Robust optimal periodic system operation)**.** System (1.5) is said to be robustly optimally operated at periodic operation with respect to the stage cost (1.6) and the constraints $x \in \mathbb{X}$ and $u \in \mathbb{U}$ if for any feasible nominal input sequence $\boldsymbol{v}(.)$ and its associated nominal state sequence $z_{\boldsymbol{v}(.)}$ it holds that

$$\liminf_{T\to\infty} \frac{\sum_{t=0}^{T} \ell^{\text{int}}(z(t), v(t))}{T} \geq \frac{1}{P}\sum_{k=0}^{P-1} \ell^{\text{int}}(z_k^p, v_k^p), \tag{4.12}$$

where $(z_k^p, v_k^p) \in \Pi$ and Π is the robust optimal periodic orbit as defined in (4.5). ■

In order to come up with a sufficient condition for optimality of a given periodic orbit Π in the sense of Def. 4.6.1, we introduce the following auxiliary result as an extension to the nominal (non-integrated) stage cost case from [19, Lem. 13]

Lemma 4.6.2 (Modified from [19])**.** *System (1.5) is robustly optimally operated at a P-periodic orbit Π if and only if the corresponding P-step system is robustly optimally operated at the steady state corresponding to Π.* ■

Proof. The proof follows along the lines of [19]. Consider $z \in \bar{\mathbb{X}}, \boldsymbol{v} \in \bar{\mathbb{U}}^{\infty}(z)$ and let $\tilde{v}(t) := (v(tP), v(tP+1), \ldots, v((t+1)P-1))$. We have

$$\liminf_{T\to\infty} \frac{\sum_{t=0}^{PT-1} \ell^{\text{int}}(z_{\boldsymbol{v}}(t,z), v(t))}{TP} = \liminf_{T\to\infty} \frac{\sum_{t=0}^{T-1} \tilde{\ell}^{\text{int}}(\tilde{z}_{\boldsymbol{v}}(t,z), \tilde{v}(t))}{TP}.$$

If the P-step system is robustly optimallyoperated at steady-state as introduced in [4, Def. 4] we get

$$\frac{1}{P} \liminf_{T\to\infty} \frac{\sum_{t=0}^{T-1} \tilde{\ell}^{\text{int}}(\tilde{z}_{\boldsymbol{v}}(t,z), \tilde{v}(t))}{T} \geq \frac{1}{P} \underbrace{\tilde{\ell}^{\text{int}}(\tilde{z}^*, \tilde{v}^*)}_{=\sum_{k=0}^{P} \ell^{\text{int}}(z_k^{p,*}, v_k^{p,*})} \tag{4.13}$$

with $(\tilde{z}^*, \tilde{v}^*) \in \tilde{\Pi}$. Further we state that

$$\liminf_{T\to\infty} \frac{\sum_{t=0}^{PT-1} \ell^{\text{int}}(z_{\boldsymbol{v}}(t,z), v(t))}{TP} = \liminf_{T\to\infty} \frac{\sum_{t=0}^{T-1} \ell^{\text{int}}(z_{\boldsymbol{v}}(t,z), v(t))}{T}. \tag{4.14}$$

Using (4.13) and (4.14) we get

$$\liminf_{T\to\infty} \frac{\sum_{t=0}^{T-1} \ell^{\text{int}}(z_{\boldsymbol{v}}(t,z), v(t))}{T} \geq \frac{1}{P} \sum_{k=0}^{P} \ell^{\text{int}}(z_k^{p*}, v_k^{p*}) \tag{4.15}$$

which means that if the P-step system is robustly optimallyoperated at steady state it follows that the corresponding system is optimally operated at the periodic orbit Π. Further, if the original system is optimally operated at the periodic orbit Π we have that (4.15) is valid and with (4.14) also (4.13) holds. Therefore it follows that in this case, the P-step system is optimally operated at the steady state $(\tilde{z}^*, \tilde{v}^*)$ corresponding to Π. Thus we have necessary and sufficient arguments for the statement postulated. It remains to show that (4.14) holds with equality. Therefore use

$$N(T^*) := P - T^* \mathrm{mod} P$$

to state

$$\liminf_{T\to\infty} \frac{\sum_{t=0}^{PT-1} \ell^{\text{int}}(z_{\boldsymbol{v}}(t,z), v(t))}{TP} = \liminf_{T^*\to\infty} \frac{\sum_{t=0}^{T^*+N(T^*)-1} \ell^{\text{int}}(z_{\boldsymbol{v}}(t,z), v(t))}{T^* + N(T^*)}$$

since the sequences consist of the same elements for $T > 0$ and $T^* > 0$. By the definition of '$\lim\inf$' it holds

$$\liminf_{T^*\to\infty} \frac{\sum_{t=0}^{T^*+N(T^*)-1} \ell^{\text{int}}(z_{\boldsymbol{v}}(t,z),v(t))}{T^*+N(T^*)} =$$

$$\lim_{\Gamma^*\to\infty}\left[\inf_{T^*\geq\Gamma^*}\left(\underbrace{\frac{\sum_{t=0}^{T^*-1} \ell^{\text{int}}(z_{\boldsymbol{v}}(t,z),v(t))}{T^*+N(T^*)}}_{\to\frac{\sum_{t=0}^{T^*-1} \ell^{\text{int}}(z_{\boldsymbol{v}}(t,z),v(t))}{T^*}} + \underbrace{\frac{\sum_{t=T^*}^{T^*+N(T^*)-1} \ell^{\text{int}}(z_{\boldsymbol{v}}(t,z),v(t)}{T^*+N(T^*)}}_{\to 0}\right)\right]$$

which shows that (4.14) is true and thus the proof is complete. □

Using Lem. 4.6.2 we are now able to give the following checkable necessary condition for robust optimally periodic operation as an extension to [4, Thm. 2].

Theorem 4.6.3. *Consider system (1.5) and let Ω be an RCI set for the associated error dynamics (4.3). If the nominal system (4.1) is dissipative with respect to the periodic orbit Π, i.e. there exists a storage function $\tilde{\lambda} : \mathbb{R}^{Pn} \to \mathbb{R}$ such that for all $\tilde{z} \in \bar{\mathbb{X}}^P, \tilde{v} \in \bar{\mathbb{U}}^P$*

$$\tilde{\lambda}(f^P(\tilde{z},\tilde{v},0)) - \tilde{\lambda}(\tilde{z}) \leq \tilde{\ell}^{\text{int}}(\tilde{z},\tilde{v}) - \sum_{k=0}^{P-1} \ell^{\text{int}}(z_k^p,v_k^p) \tag{4.16}$$

and $(z_k^p,v_k^p) \in \Pi$, $k \in \mathcal{I}_{[0,P-1]}$, then system (1.5) is robustly optimally operated at the periodic orbit Π. ■

Proof. The proof is adapted from [3]. Consider $z \in \mathbb{X}, \boldsymbol{v} \in \mathbb{U}^\infty(z)$. Taking the average of both sides of (4.16) then yields

$$\begin{aligned}
&\liminf_{T\to\infty} \frac{\sum_{t=0}^{T-1}(\tilde{\lambda}(\tilde{z}_{\boldsymbol{v}}(tP+P,z)) - \lambda(\tilde{z}_{\boldsymbol{v}}(tP,z)))}{T} \\
&\qquad\qquad \leq \liminf_{t\to\infty} \frac{\sum_{t=0}^{T-1} \tilde{\ell}^{\mathrm{int}}(\tilde{z}_{\boldsymbol{v}}(tP,z), \tilde{v}(tP))}{T} - \sum_{k=0}^{P-1} \ell^{\mathrm{int}}(z_k^p, v_k^p) \\
&\Leftrightarrow \\
&\underbrace{-\liminf_{T\to\infty} \frac{\sum_{t=0}^{T-1}(\tilde{\lambda}(\tilde{z}_{\boldsymbol{v}}(tP+P,z)) - \lambda(\tilde{z}_{\boldsymbol{v}}(tP,z)))}{T}}_{=:(\star)} \\
&\qquad\qquad + \liminf_{t\to\infty} \frac{\sum_{t=0}^{T-1} \tilde{\ell}^{\mathrm{int}}(\tilde{z}_{\boldsymbol{v}}(tP,z), \tilde{v}(tP))}{T} \geq \sum_{k=0}^{P-1} \ell^{\mathrm{int}}(z_k^p, v_k^p).
\end{aligned}$$

By investigating $(\star)$ we note the telescoping series and assume w.l.o.g.[1] $\tilde{\lambda}(\tilde{z}) \geq 0$ for all $\tilde{z} \in \mathbb{X}$ and get

$$\liminf_{T\to\infty} \frac{\tilde{\lambda}(\tilde{z}_{\boldsymbol{v}}(tP,z)) - \lambda(\tilde{z}_{\boldsymbol{v}}(0))}{T} \geq \lim_{\Gamma\to\infty} \left(\inf_{T\geq\Gamma} -\frac{\tilde{\lambda}(\tilde{z}_{\boldsymbol{v}}(0))}{T} \right) = 0.$$

Therefore we can conclude that $(\star) \geq 0$ which completes the proof. □

Definition 4.6.4. The nominal system is called robustly strict dissipative with respect to the robust optimal periodic orbit Π if there exists a storage function $\tilde{\lambda} : \mathbb{R}^{Pn} \to \mathbb{R}$ and in addition there exists a function $\alpha \in \mathcal{K}_\infty$ such that

$$\tilde{\lambda}(f^P(\tilde{z}, \tilde{v}, 0)) - \tilde{\lambda}(\tilde{z}) \leq \tilde{\ell}^{\mathrm{int}}(\tilde{z}, \tilde{v}) - \sum_{k=0}^{P-1} \ell^{\mathrm{int}}(z_k^p, v_k^p) - \alpha(|(\tilde{z}, \tilde{v})|_\Pi). \quad (4.17)$$

■

4.7 Stability analysis

In this section we investigate stability by constructing a Lyapunov function based on previous assumptions. The proof for stability of the nominal system is conceptually related to [1] and [4].

[1] Because $\mathbb{X}$ is compact.

Assumption 4.7.1. *The nominal system is robustly strict dissipative with respect to* Π *according to Def. 4.6.4 with a continuous storage function* $\tilde{\lambda}$. ■

Assumption 4.7.2. *For all* $z(0) \in \Pi_{\mathbb{X}}$ *the solution* $\boldsymbol{v}^*$ *of* $(P_{\mathit{REMPC\text{-}P}})$ *is element of* $\tilde{\Pi}_{\mathbb{U}}^{N/P}$ *such that for the corresponding states it holds* $\tilde{z}_{\boldsymbol{v}^*}(k) \in \tilde{\Pi}_{\mathbb{X}}$ *for all* $k \in \mathcal{I}_{[P-1,N]}$. ■

Compare with Rem. 3.5.3. Consider the set

$$Z_N := \{x \in \bar{\mathbb{X}} | \exists \boldsymbol{v} \in \bar{\mathbb{U}}^N(z) \text{ s.t. } z_{\boldsymbol{v}}(N,x) \in \bar{\mathbb{X}}_f\} \tag{4.18}$$

for which $(P_{\text{REMPC-P}})$ is feasible and

$$\tilde{Z}_N := \{\tilde{z} \in \bar{\mathbb{X}}^P | z_{P-1} \in Z_N\}.$$

Corollary 4.7.3 (Asymptotic stability of nominal system)**.** *If Ass. 4.3.1, 4.4.1, 4.4.3, 4.7.1, and 4.7.2 are fulfilled, then under application of Alg. 3 the* P*-step system according to the nominal system* (4.9) *is asymptotic stable w.r.t. the set* $\tilde{\Pi}_{\mathbb{X}}$ *with region of attraction*[2] $\tilde{Z}_N$. ■

Proof. The proof follows directly by choosing $x := z$, $u := v$, $\mathbb{X} := \bar{\mathbb{X}}$, $\mathbb{U} := \bar{\mathbb{U}}$,$\ell := \ell^{\text{int}}$ and $V_f := \bar{V}_f$ from Thm. 3.5.13 □

By using the typical argument in tube-based robust MPC [20] and particularly in the sense of [4, Thm. 4] we state stability of the composition of the nominal and real P-step system. Therefore note, that the RCI set for the P-step system is given by $\Omega^P \subseteq \mathbb{R}^{nP}$.

Theorem 4.7.4 (Asymptotic stability of composite system)**.** *If Ass. 4.3.1, 4.4.1, 4.4.3, 4.7.1, and 4.7.2 are fulfilled, then under application of Alg. 3, the set* $\mathcal{A} := \tilde{\Pi}_{\mathbb{X}} \times \tilde{\Pi}_{\mathbb{X}} \oplus \Omega^P$ *is asymptotically stable for the composition of the* P*-step systems according to* (4.1) *and* (1.5)*. The region of attraction is* $\tilde{Z}_N \times \tilde{Z}_N \oplus \Omega^P$.
■

Proof. By Cor. 4.7.3 there exists a $\mathcal{KL}$ function β such that $|\tilde{z}_{\text{MPC}}(tP)|_{\tilde{\Pi}_{\mathbb{X}}} \leq \beta(|\tilde{z}(0)|_{\tilde{\Pi}_{\mathbb{X}}}, tP)$. As $\tilde{x}_{\text{MPC}}(tP) = \tilde{z}_{\text{MPC}}(tP) + \tilde{e}_{\text{MPC}}(tP)$ and $\tilde{e}_{\text{MPC}}(tP) \in \Omega^P$ for all $t \in \mathcal{I}_{\geq 0}$ we calculate

$$\begin{aligned} |\tilde{x}_{\text{MPC}}(tP)|_{\tilde{\Pi}_{\mathbb{X}} \oplus \Omega^P} &= |\tilde{z}_{\text{MPC}}(tP) + \tilde{e}_{\text{MPC}}(tP)|_{\tilde{\Pi}_{\mathbb{X}} \oplus \Omega^P} \\ &\leq |\tilde{z}_{\text{MPC}}(tP)|_{\tilde{\Pi}_{\mathbb{X}}} \\ &\leq \beta(|\tilde{z}(0)|_{\tilde{\Pi}_{\mathbb{X}}}, tP). \end{aligned}$$

[2] The region of attraction is the set from which we can reach the terminal region with respect to the last nominal state of the P-step system state.

Using this, for all $\forall t \in \mathcal{I}_{\geq 0}$

$$\begin{aligned}|(\tilde{z}_{\text{MPC}}(tP),\tilde{x}_{\text{MPC}}(tP)|_{\mathcal{A}} &= |\tilde{z}_{\text{MPC}}(tP)|_{\tilde{\Pi}_{\mathbb{X}}} + |\tilde{x}_{\text{MPC}}(tP)|_{\tilde{\Pi}_{\mathbb{X}}\oplus\Omega^P} \\ &\leq 2\beta(|\tilde{z}_{\text{MPC}}(0)|_{\tilde{\Pi}_{\mathbb{X}}},tP) \\ &\leq 2\beta(|\tilde{z}_{\text{MPC}}(0),\tilde{x}_{\text{MPC}}(0))|_{\mathcal{A}},tP).\end{aligned}$$

This proves that the set $\mathcal{A}$ is asymptotically stable for the composite system with region of attraction $\tilde{Z}_T \times \tilde{Z}_T \oplus \Omega^P$. □

Remark 4.7.5. *As mentioned in remark 3.5.14, we do not have asymptotic stability of the composition of the nominal system and the real system but of the composition of the respective P-step systems.* ■

Robust asymptotic convergence of the real system

Thm. 4.7.4 only shows stability of the composite system (4.1) and (1.5), but not asymptotic stability of the P-step system according to the real closed-loop system (1.5). Let

$$\tilde{z}(P-1) = \tilde{x}(P-1) \tag{4.19}$$

From stability of the nominal system we conclude that there exists a Class $\mathcal{KL}$ function β such that $|\tilde{z}(tP)|_{\tilde{\Pi}_{\mathbb{X}}} \leq \beta(|\tilde{z}(0)|_{\tilde{\Pi}_{\mathbb{X}}},tP)$. Using $\tilde{z}(P-1) = \tilde{x}(P-1)$ we have

$$|\tilde{x}(tP-1)|_{\tilde{\Pi}_{\mathbb{X}}\oplus\Omega^P} \leq \beta(|\tilde{z}(P-1)|_{\tilde{\Pi}_{\mathbb{X}}},tP) \tag{4.20}$$

and therefore the real P-step system state $\tilde{x}$ (robustly) converges to $\tilde{\Pi}_{\mathbb{X}} \oplus \Omega^P$ by choosing the first P states of z according to (4.19). This does not imply Lyapunov stability of $\tilde{x}$ w.r.t. $\tilde{\Pi}_{\mathbb{X}} \oplus \Omega^P$, as e.g. discussed in [20, p. 236].

4.8 Outline: Tube-based robust economic MPC without terminal constraints

Without any rigurous statement or proof we propose another tube-based robust economic MPC method by combining the tube-based concept introduced in the previous chapter with the terminal set and cost free method [18], introduced in Sec. 3.6.

Let $N \in \mathcal{I}_{>0}$ and consider the nominal open loop optimization problem

$$(P_{\text{REMPCU-P}}) \begin{cases} \min_{\boldsymbol{v}\in\bar{\mathbb{U}}^N} \sum_{k=0}^{N-1} \ell^{\text{int}}(z_{\boldsymbol{v}}(k,z), v(k)) \\ \text{s.t. for all } k \in \mathcal{I}_{[0,N-1]} : \\ \qquad z_{\boldsymbol{v}}(k+1,z) = f(z_{\boldsymbol{v}}(k,z), v(k), 0) \\ \qquad z_{\boldsymbol{v}}(k,z) \in \bar{\mathbb{X}} \\ \qquad v(k) \in \bar{\mathbb{U}} \\ \qquad z_{\boldsymbol{v}}(0,z) = z \end{cases}$$

We propose Alg. 4 and state the conjecture, that under similar assumptions with respect to the nominal system and tightened constraints as in Sec. 3.6 as well as Ass. 4.3.1 one can establish similar performance and stability results as in the original work [18], but for the composite system with respect to the set induced by the tube.

4.9 Example: Simple supply chain network

Consider the simple supply chain example introduced in Sec. 1.3 with nominal disturbances (1.14) and unknown disturbances (3.20). Define for easier notation in this section

$$x := [x_{S,1}, x_{S,2}, x_{T,P}, x_{T,L}, x_R]^\top.$$

Algorithm 4 Robust economic model predictive control without terminal constraints for optimal periodic operation

1: **procedure** REMPCU-P(initial state $x(0)$)
2:     **for** $k_1 = 0, P, 2P, ..$ **do**
3:         solve $(P_{\text{REMPCU-P}})$
4:         **for** $k_2 = 0,1,..,P-1$ **do**
5:             $u_{\text{MPC}}(k_1+k_2) = \phi(v^*(k_1+k_2), x_{\text{MPC}}(k_1+k_2), z_{\text{MPC}}(k_1+k_2))$
6:             $v_{\text{MPC}}(k_1+k_2) = v^*(k_1+k_2)$
7:         **end for**
8:     **end for**
9: **end procedure**

Robust control invariant set

In order to construct the robust control invariant set (Def. 4.2.2), we introduce an auxiliary controller. Let $e_T := x_{T,L} - z_{T,L}$, $e_R := x_R - z_R$, and

$$\phi(v,x,z) = \begin{cases} [v_{T,L} - e_T, v_S]^\top, x_{T,P} = 0 \\ [v_{T,L} + e_R, v_S - e_R]^\top, x_{T,P} = 1. \end{cases} \tag{4.21}$$

We tighten the truck load constraints to $4 \leq z_T \leq 10$ and restrict the truck position to be at the retailer or to follow the optimal orbit for two consecutive time instances[3]. A valid trajectory for the truck position is e.g. supplier, retailer, supplier, supplier, supplier, retailer. The control law 4.21 can be understood as follows. As the truck arrives at the retail store, it unloads the nominal amount $v_{T,L}$ plus the amount of goods which is missing because of the unknown disturbance e_R. Since we tightened the nominal constraint set, we can guarantee that worst-case demands over one period (4 items) can be handled. Also, we trigger an additional production of items, at the same time as the truck unloads at the retailer for disturbance compensation. In the next time instance, the truck is at the supplier and loads the nominal amount of goods, plus the amount of goods that are missing in the trucks storage because of disturbance compensation at the previous time instance, when the truck was at the retailer storage.

Application of (4.21) leads to the RCI set

$$\Omega = \left\{ x \in \mathbb{R}^5 \text{ s.t.} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -4 \\ -4 \end{bmatrix} \leq x \leq \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}. \tag{4.22}$$

Note that the RCI set of $x_{S,2}$ is empty, because by (4.21) any additional production is directly loaded into the truck without storing them. The corresponding

[3] This is an extension to the concept of the previous chapter, since we provide a tightened constraint set for the 2-step graph state.

reduced nominal feasible set $\bar{\mathbb{X}} = \mathbb{X} \ominus \Omega$ with $a \in \mathbb{R}, a > 100$ is given by

$$
\begin{aligned}
0 &\leq x_{S,1} \leq a \\
0 &\leq x_{S,2} \leq a \\
4 &\leq x_{T,L} \leq 10 \\
-a &\leq x_R \leq a
\end{aligned}
$$

as well as tightened graph constraints for the truck position explained in the beginning of this paragraph.

Robust optimal periodic operation

Define $\eta_1(x) := x_{S,1} + x_{T,L} + x_R$ and $\eta_2(x) := x_{S,1} + x_{T,L} - 10x_R$. Using the RCI set (4.22), the integrated stage cost can be calculated as

$$
\begin{aligned}
\ell^{\text{int}}(z,v) &= 0.5 z_{S,2} + v_{T,P} \\
&+ \begin{cases}
\int_{z_{S,1}}^{z_{S,1}+4} \left(\int_{z_{T,L}-4}^{z_{T,L}} \left(\int_{z_R-4}^{0} \eta_1(x) dx_R \right) dx_{T,L} \right) dx_{S,1}, & \text{for } z_R \geq 4 \\
\int_{z_{S,1}}^{z_{S,1}+4} \left(\int_{z_{T,L}-4}^{z_{T,L}} \left(\int_{z_R-4}^{0} \eta_1(x) dx_R + \int_0^{z_R} \eta_2(x) dx_R \right) dx_{T,L} \right) dx_{S,1}, & \\
\qquad \text{for } 0 < z_R < 4 & \\
\int_{z_{S,1}}^{z_{S,1}+4} \left(\int_{z_{T,L}-4}^{z_{T,L}} \left(\int_{z_R-4}^{z_R} \eta_1(x) dx_R \right) dx_{T,L} \right) dx_{S,1}, & \text{for } z_R \leq 0
\end{cases}
\end{aligned}
\tag{4.23}
$$

by Fubinis Theorem [9]. *Importantly*, note that we used the original cost functional for $v_{T,P}$ and $z_{S,2}$ since the case of an empty RCI set is neither covered in the original work [4] nor explicitly considered here. As a conjecture we think, that simply adding the nominal stage cost in the case of an empty RCI set dimension is not the best treatment in general. However, given our specific problem structure, it turns out to work well in practice, as the experiments show. Because of symmetry, the integrals over $x_{S,1}$ and $x_{T,L}$ yield linear functions again. However, in the second case of (4.23), we get a quadratic functional depending on z_R. As a consequence we can not apply the explicit method for verifying strict dissipativity for a given orbit, which was derived in Sec. 2.3.

Solving (4.5) using the RCI set (4.22) yields

$$\bar{P} = 2, \tag{4.24}$$

$$\bar{\Pi} \approx \left\{ \left(\begin{bmatrix} 2 \\ 0 \\ 0 \\ 4 \\ 4.2727 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right), \left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 6 \\ 3.2727 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \right) \right\} \tag{4.25}$$

with average integrated stage cost $\frac{1}{2}\sum_{k=0}^{1} \ell^{\text{int}}(\bar{x}_k^p, \bar{u}_k^p) \approx 776.75$ and average nominal stage cost $\frac{1}{2}\sum_{k=0}^{1} \ell(\bar{x}_k^p, \bar{u}_k^p) \approx 10.77$ along the periodic orbit. Because of the convex objective and linear constraints for the fixed truck position periodic orbit $(0,1)$ we have by the weak slater condition strong duality, and therefore by Thm. 2.3.3 dissipativity. As mentioned above, we can not explicitely verify strict dissipatvity using uniqueness of a certain linear program, because of the quadratic problem structure. Instead, we verified strict dissipativity experimentally by many (10000) uniformly random chosen starting points for the optimization problem (4.5). Therefore we used the Matlab Global Optimization Toolbox, using a local, primal dual optimization algorithm with several restarts. For each run, we obtained the same optimizer and therefore we conclude (experimental) strict dissipativity by Thm. 2.3.5. Again, we only showed strict dissipativity for the fixed truck's position periodic orbit $(0,1)$. Because the storage values of the retailer's and truck's storage are just shifted w.r.t. the nominal optimal orbit (1.20), strict dissipativity can be handled as it was done in Sec. 2.5.

Nominal terminal cost and terminal set

We modify the terminal set and cost, presented in Sec. 3.7 w.r.t. to the integrated stage cost function. Therefore in order to fulfill Ass. 4.4.1 and Ass. 4.7.2 choose the terminal set

$$\bar{X}_f := \left\{ \tilde{x} \in \{\bar{\mathbb{X}} \cap \{ \left\{ \begin{matrix} x_{S,1} = 0 \\ x_{S,2} = 0 \\ x_{T,P} = 1 \\ x_{T,L} = 6 - x_R \\ x_R \leq 0 \end{matrix} \right\} \cup \left\{ \begin{matrix} x_{S,1} = 2 \\ x_{S,2} = 0 \\ x_{T,P} = 0 \\ x_{T,L} = 4 \\ x_R = 4.2727 \end{matrix} \right\} \}\} \right\}.$$

The terminal controller summarizes two consecutive inputs of the original system ($P = 2$). Let

$$\tilde{\kappa}_f^0 := \begin{bmatrix} 1 \\ x_R - 3.2727 \\ 3.2727 - x_R + 2 - x_{T,L} + 4 - x_{S,2} + 4 \end{bmatrix}$$

and define with $x^+ := f(\tilde{x}_0, \tilde{\kappa}_f^0, 0)$

$$\tilde{\kappa}_f^1 := \begin{bmatrix} 1 \\ \min\{x_{S,2}^+, 4.2727 - x_R^+\} \\ 0 \end{bmatrix}$$

and finally

$$\tilde{\kappa} := \begin{cases} (\tilde{\kappa}_f^{0,\top}, \tilde{\kappa}_f^{1,\top}), \, x_{T,P} = 1 \\ ([1,-2,2]^\top, [1,2,0]^\top), \, x_{T,P} = 0. \end{cases}$$

As terminal cost choose

$$\tilde{V}_f := x_{T,P} 475(3.2727 - x_R).$$

The terminal configuration is a straight-forward adaption of the one presented in Sec. 3.7 for the shifted optimal orbit w.r.t. the retail store values, the integrated stage cost function and the tightened constraints.

Results

Using the example of the simple supply chain system from Sec. 1.3, we demonstrate the capabilites and performance of the proposed control scheme (Alg. 3) and compare it with a 'robustified' version of an existing control scheme (Alg. 4). In Fig. 4.1 and Fig. 4.2 we show sample closed loop simulations. For Alg. 3 a planning horizon $N = 4$ is used and for Alg. 4, $N = 6$, which still yields feasible computation times of the mixed-integer online optimization problem. Note, that in case of a terminal constraint and cost, the distance to the optimal periodic orbit is significantly smaller, compared to the case without terminal constraint and terminal cost. This leads to a better asymptotic average performance of the proposed algorithm as shown in Tab. 4.1. We need to choose a planning horizon $N = 6$ for the unconstrained case in order to get a similar performance of Alg. 4 compared to the Alg. 3. This is related to the nominal closed loop results shown in

Fig. 3.2, in which (for a special initial condition) we need a planning horizon $N = 6$ in order to achieve the optimal asymptotic average performance. *Importantly* note, that both algorithms achieve a better performance than the robust optimal operation.

Compared to the asymptotic average performances of the nominal control schemes applied in case of disturbances (Tab. 3.2), we have in case of Alg. 3 only a *quarter* of the cost within the same planning horizon $N = 4$. The improvement under application of Alg. 4 for $N = 6$ is in the same range as well. Interestingly, since we choose a different initial condition for the robust setting and since we have tightened constraints, using the planning horizon $N = 4$ already leads to unstable closed loop behavior of the nominal and therefore also for the real system. Choosing $N = 5$ provides stable, but suboptimal performance. As mentioned in the previous chapter, the computation times for $N = 6$ are essentially worse compared to those using $N = 4$ in the constraint terminal set and terminal cost case.

Therefore we conclude that the concept of the robust stage cost [4], extended to the periodic operation case yields significant performance improvements for constrained and unconstrained underlying EMPC algorithms for periodic operation.

Table 4.1: Comparison of Alg. 3 and Alg. 4 applied to the simple supply chain model under disturbances after 4000 simulation steps.

Algorithm	Planning hor. N	Avg. Performance
Alg. 3	4	9.72
Alg. 4	4	$\rightarrow \infty$
Alg. 4	5	10.67
Alg. 4	6	9.85

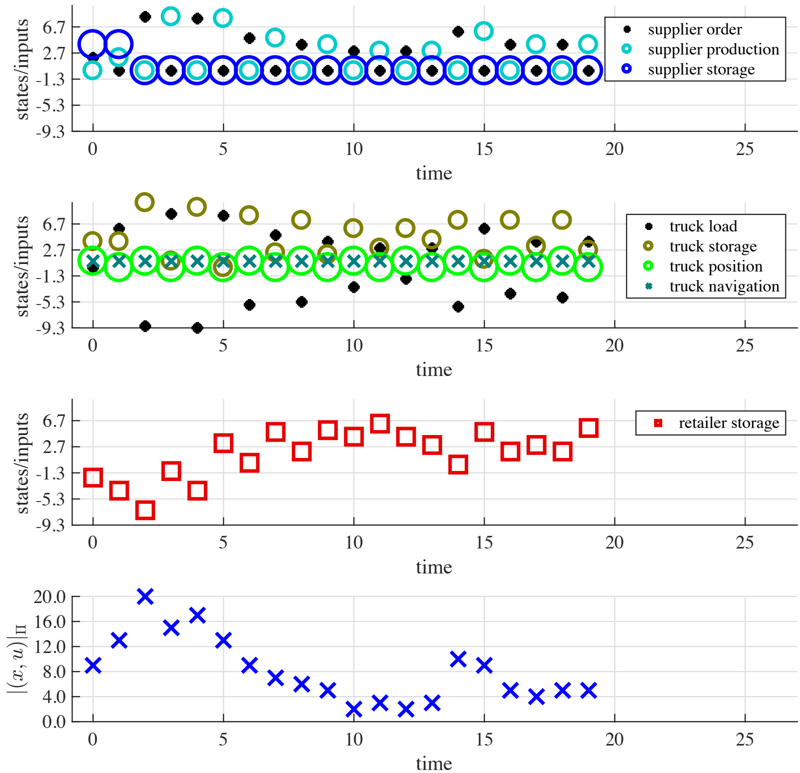

Figure 4.1: Simulation of the closed loop system with planning horizon $N = 4$ under application of Alg. 3.

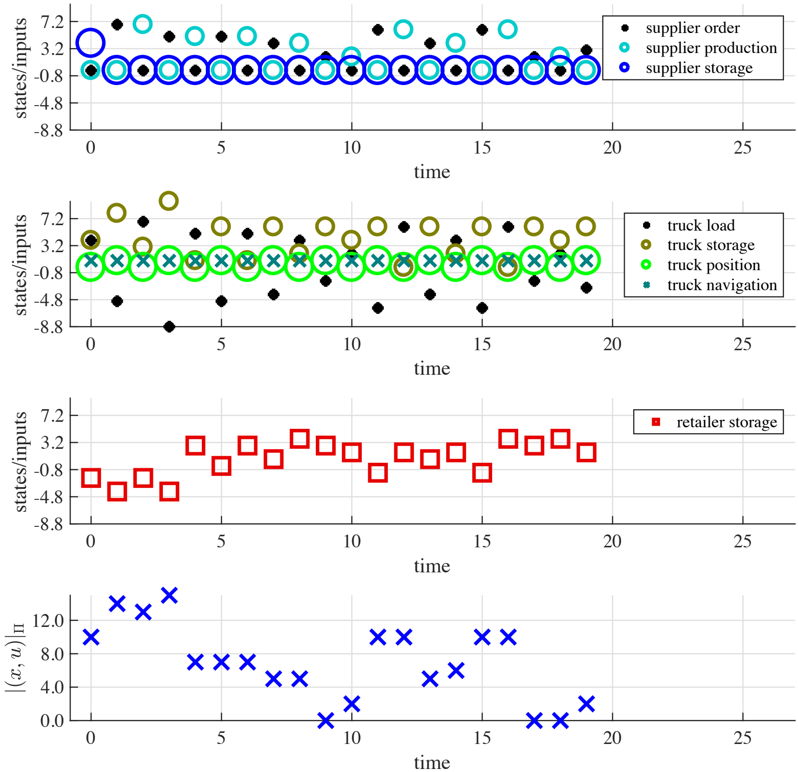

Figure 4.2: Closed loop results of the simple supply chain network using Alg. 4 and planning horizon $N = 6$.

5 Application: Complex supply chain network

5.1 Introduction

Typical supply chain networks consist of suppliers, manufacturers, a distribution network and customers. In Fig. 5.1, an example supply chain network without auxiliary suppliers is shown. It consists of a supplier for one type of good and network graph on which one truck distributes the goods to three retail stores. Note that this model is essentially more complex than the simple supply chain introduced in Sec. 1.3. Mainly because of the more complex network structure which is particularly hard to handle within the upcoming online optimization problem.

Despite simplifications, the model class which contains the supply chain network (Fig. 5.1) and the resulting mixed integer problem represents the most frequently used class of models and resulting optimization problems in literature [17].

In most supply chain planning publications an open loop, finite time horizon optimization is performed in order to minimize the overall cost. In more recent publications, e.g. in [16] or [22], model predictive control concepts are used in order to improve long term performance. However those approaches are either heuristic or they are designed for tracking a reference orbit. In both cases, it is not possible to state any performance guarantees. In [14] unknown disturbances are explicitly taken into account by formulating a linear stochastic optimization problem for performance improvement.

All of the afore mentioned approaches lack strict theoretical foundation in terms of recursive feasibility, asymptotic average performance and convergence properties with respect to the optimal orbit. To this end, in [21] a strict economic model predictive control scheme for inventory management in supply chains is presented, that provides the theoretical properties of recursive feasibility, asymptotic average performance of the optimal steady state and asymptotic convergent behavior to the optimal steady state. However only the case of optimal steady state without external unknown disturbances is considered.

In this chapter we overcome the limitations of existing literature regarding model predictive control and supply chain networks in terms of

- *Verification* of a given periodic orbit in terms of optimal operation of the closed loop system
- Economic model predictive control, in case *periodic* operation is optimal
- *Robustness* and better closed loop performance under the presence of *disturbances*.

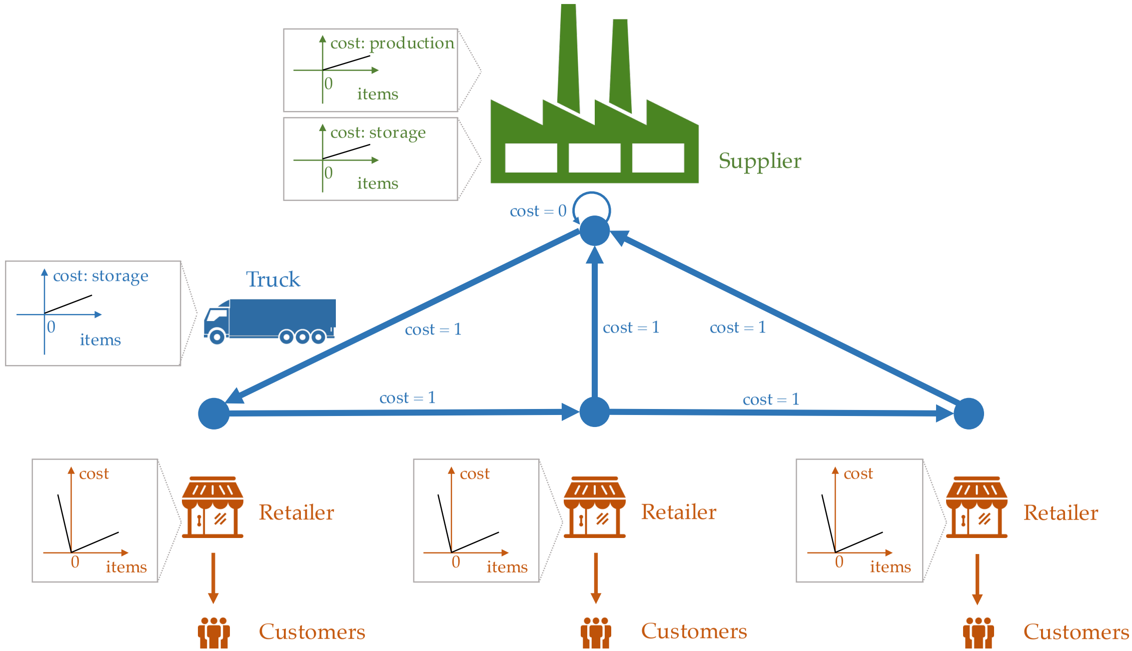

Figure 5.1: Elements of the supply chain network. Illustrated are the qualitative cost functions and the graph on which the truck is able to drive.

5.2 Model

Dynamics

Like in the introductory example (Sec. 1.3), we use the following notations: $x_{S,1} \in \mathbb{R}$ represents the number of goods in the supplier production process, $x_{S,2} \in \mathbb{R}$ the number of goods in the supplier storage, $x_{T,P} \in \{0,1\}$ describes the truck position, $x_{T,L} \in \mathbb{R}$ the number of goods which are carried by the truck and $x_{R,1}, x_{R,2}, x_{R,3} \in \mathbb{R}$ the number of goods in the retailers storages. As well as inputs, namely the truck navigation $u_{T,P} \in \{0,1\}$ and truck load of goods $u_{T,L} \in \mathbb{R}$, supplier production request (number of goods) $u_S \in \mathbb{R}$ and external disturbance (number of goods) $w \in \mathbb{W}$ where $w(k) = w^* + \epsilon$ with $\epsilon \sim Y_\epsilon$ with probability distribution $P(\epsilon)$ that describes the costumer demand at the retail store. We assume $\mathbb{E}[\epsilon] = 0$ from which follows that we have $\mathbb{E}[w(k)] = w^*$. The corresponding switched system dynamics is given as

$$
\begin{aligned}
\begin{bmatrix} x_{S,1}(k+1) \\ x_{S,2}(k+1) \\ x_{T,P}(k+1) \\ x_{T,L}(k+1) \\ x_{R,1}(k+1) \\ x_{R,2}(k+1) \\ x_{R,3}(k+1) \end{bmatrix}
&= \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{=:A}
\underbrace{\begin{bmatrix} x_{S,1}(k) \\ x_{S,2}(k) \\ x_{T,P}(k) \\ x_{T,L}(k) \\ x_{R,1}(k) \\ x_{R,2}(k) \\ x_{R,3}(k) \end{bmatrix}}_{=:x(k)} \\
&\quad + \underbrace{\begin{bmatrix} 0 \\ 0 \\ f_{T,P}(x_{T,P}(k), u_{T,P}(k)) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{=:f_G(x(k),u(k))} + B_{\sigma(k)} \underbrace{\begin{bmatrix} u_{T,L}(k) \\ u_S(k) \end{bmatrix}}_{=:u_B(k)} + w(k)
\end{aligned}
\tag{5.1}
$$

with states $x(k)$, inputs $u(k) = [u_{T,P}(k), u_{T,L}(k), u_S(k)]^\top$ and nominal disturbance

$$
w^*(k) = [0,0,0,0,-1,-1,-1]^\top \tag{5.2}
$$

as well as unknown, uniformly distributed disturbances

$$w(k) \in \left\{ w \in \mathcal{I}^7 \text{ s.t. } \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -2 \\ -2 \\ -2 \end{bmatrix} \leq w \leq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ -1 \\ -1 \end{bmatrix} \right\}. \tag{5.3}$$

The switched input matrix $B_{\sigma(k)}$ is defined as

$$B_{\sigma(k)} \in \{B_0, B_1, B_2, B_3\}, \quad B_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$
$$B_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \end{bmatrix}, B_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ -1 & 0 \end{bmatrix},$$

together with the switching policy $\sigma(k) := x_{T,P}(k)$. The dynamics f_G of the truck are encoded in a Graph, see Fig. 5.1. The graph encodes the supply network structure and the graph dynamics describe how the truck can travel. In short we have dynamics of the form

$$x(k+1) = Ax(k) + B_{\sigma(k)} u_B(k) + f_G(x(k), u(k)) + w(k) \tag{5.4}$$

with $x \in \mathbb{X}$ and $u \in \mathbb{U}$. In addition with $a \in \mathbb{R}, a > 100$ we have the following state and input constraints

$$\begin{aligned}
0 &\leq x_{S,1} \leq a \\
0 &\leq x_{S,2} \leq a \\
0 &\leq x_{T,L} \leq 30 \\
-a &\leq x_R \leq a \\
-a &\leq x_R \leq a \\
-a &\leq x_R \leq a.
\end{aligned} \tag{5.5}$$

Stage cost

The stage cost is similarly defined as in the case of the simple supply chain example, introduced in Sec. 1.3.

1. The production and storage cost is defined by $\ell_S(x,u) := x_{S,1} + 0.5x_{S,2}$.
2. The cost for truck load and driving reads $\ell_T(x,u) := x_{T,L} + u_{T,P}$.
3. For each retail store we have a storage cost for a positive number of goods in the store. A larger demand than available goods (negative number of goods), which results in unhappiness of the customers, is modelled by a high cost. More precisely for the stores $i = 1,2,3$ we have

$$\ell_{R,i}(x,u) := \begin{cases} -10x_{R,i}, & x_{R,i} < 0 \\ x_{R,i}, & x_{R,i} \geq 0. \end{cases} \tag{5.6}$$

In summary we end up with a piece-wise defined linear stage cost, defined on 9 disjoint subsets (i.e. $x_{R,1} \geq 0, x_{R,2} \geq 0, x_{R,3} \geq 0$, or $x_{R,1} < 0, x_{R,2} \geq 0, x_{R,3} \geq 0$). The stage cost is continuous and bounded on $\mathbb{X} \times \mathbb{U}$.

5.3 Optimal operation

Nominal optimal periodic orbit

Again, we solve for the optimal periodic orbit as described in Rem. 1.2.2 approximately in terms of finite P using the dynamics (5.4), stage cost (5.6) and the expected (nominal) disturbance $w^*(k) = [0,0,0,0,-1,-1,-1]^T$. We obtain

$$\bar{P} = 4,$$
$$\bar{\Pi} =$$
$$\left\{ \left(\begin{bmatrix} 12 \\ 0 \\ 0 \\ 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 12 \\ 0 \end{bmatrix} \right), \left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 12 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix} \right), \left(\begin{bmatrix} 0 \\ 0 \\ 2 \\ 8 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix} \right), \left(\begin{bmatrix} 0 \\ 0 \\ 3 \\ 4 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 12 \end{bmatrix} \right) \right\} \tag{5.7}$$

with average cost $\frac{1}{4}\sum_{k=0}^{3} \ell(\bar{x}_k^p, \bar{u}_k^p) = 14.5$ along the periodic orbit. For a fixed time horizon, we can argument as in Sec. 2.5 that the optimal position trajectory for the truck is given by $(x_{T,P}(tP), x_{T,P}(tP+1), x_{T,P}(tP+2), x_{T,P}(tP+3)) = (0,1,2,3)$. In case of any other trajectory we would have to store additional goods, either in the truck's or in the retailer's storage in order to prevent negative retail store values because of missing supply by the truck and the nominal demands at the retail stores. Because of the dynamic constraints of the truck position, saving the 'truck is driving cost' is always higher than increasing the total number of goods in the supply chain in a previous roundtrip of the truck. By fixing the truck's trajectory, the P-step system stage cost consists of $9^4 = 6561$ different disjoint local subsets. By using Cor. 2.4.5 we verify that (5.7) is strictly dissipative (in combination with suboptimal operation for any other truck orbit and Rem. 2.2.6), and therefore the system (5.4) is optimally operated at (5.7). Furthermore by Cor. 2.2.5 it is suboptimally operated off that specific orbit.

Robust optimal periodic orbit

As in the example of the simple supply chain model in Sec. 4.9 we start by introducing an auxiliary controller in order to construct the tube. We can directly extend the auxiliary controller given in (4.21). Therefore define $e_T := x_{T,L} - z_{T,L}$, $e_{R,1} := x_{R,1} - z_{R,1}$, $e_{R,2} := x_{R,2} - z_{R,2}$, $e_{R,3} := x_{R,3} - z_{R,3}$

and

$$\phi(v,x,z) = \begin{cases} [v_{T,L} - e_T, v_S]^\top, x_{T,P} = 0 \\ [v_{T,L} + e_{R,1}, v_S - e_{R,1}]^\top, x_{T,P} = 1 \\ [v_{T,L} + e_{R,2}, v_S - e_{R,2}]^\top, x_{T,P} = 2 \\ [v_{T,L} + e_{R,3}, v_S - e_{R,3}]^\top, x_{T,P} = 3. \end{cases} \tag{5.8}$$

We tighten the truck load constraints to $12 \leq z_T \leq 30$ and in addition we restrict the truck position to follow the optimal orbit for four consecutive time instances[1]. As the truck arrives at a retail store, it unloads the nominal amount $v_{T,L}$ plus the amount of goods which is missing because of the unknown disturbance e_R. Since we tightened the nominal constraint set, we can guarantee that worst-case demands over one period (12 items) can be handled. Also, we trigger an additional production of items, at the same time as the truck unloads at the retailer for disturbance compensation.

If the truck is at the supplier it loads the nominal amount of goods, plus the amount of goods that are missing in the trucks storage because of disturbance compensation at the previous time instance, when the truck was at the retailer storage.

Application of 5.8 yields the RCI set

$$\Omega = \left\{ x \in \mathbb{R}^7 \text{ s.t. } \begin{bmatrix} 0 \\ 0 \\ 0 \\ -12 \\ -4 \\ -4 \\ -4 \end{bmatrix} \leq x \leq \begin{bmatrix} 4 \\ 9 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}. \tag{5.9}$$

Note that the RCI set of $x_{S,1}$ no longer empty as it was the case in (4.22). The corresponding reduced nominal feasible set $\bar{\mathbb{X}} = \mathbb{X} \ominus \Omega$ is given by

$$\begin{aligned} 0 \leq x_{S,1} \leq a, &\qquad 0 \leq x_{S,2} \leq a \\ 12 \leq x_{T,L} \leq 30, &\qquad -a \leq x_{R,1} \leq a \\ -a \leq x_{R,2} \leq a, &\qquad -a \leq x_{R,3} \leq a \end{aligned} \tag{5.10}$$

[1]Again, this is an extension to the concept in the previous chapter, since we provide a tightened constraint set for the 4-step graph state.

as well as constraints of the trucks position to the optimal periodic orbit (5.7).

Using the afore derived RCI set, the resulting integrated stage cost function gets very tedious. This is due to the fact, that we have to distinguish the 9 different cases of the original stage cost function. For each case, there will be 7 nested integrals by leveraging Fubinis Theorem [9]. In Addition for each integral over $x_{R,i}, i = 1{,}2{,}3$ we have to distuinguish three different cases, as we did in (4.23). Therefore, explicely writing down the integrated stage cost function does not make much sense[2]. Solving (4.5) yields

$$
\begin{aligned}
\bar{P} &= 4,\\
\bar{\Pi} &= \{\\
&\left(\begin{bmatrix}12\\0\\0\\12\\3.77\\4.77\\5.77\end{bmatrix}, \begin{bmatrix}1\\12\\0\end{bmatrix}\right), \left(\begin{bmatrix}0\\0\\1\\24\\2.77\\3.77\\4.77\end{bmatrix}, \begin{bmatrix}1\\-4\\0\end{bmatrix}\right),\\
&\left(\begin{bmatrix}0\\0\\2\\20\\5.77\\2.77\\3.77\end{bmatrix}, \begin{bmatrix}1\\-4\\0\end{bmatrix}\right), \left(\begin{bmatrix}0\\0\\3\\16\\4.77\\5.77\\2.77\end{bmatrix}, \begin{bmatrix}1\\-4\\12\end{bmatrix}\right)\\
&\}.
\end{aligned} \tag{5.11}
$$

Since we have in (5.11) exactly the same inputs as in (5.7), the integrated stage cost function simply yields a shift of the optimal trajectory in the retail store values. This is no suprise, as the cost increases dramatically, as soon as we have negative retail store values. Like it was the case in the simple supply chain example in Sec. 4.9, it is no longer possible to exactly verify strict dissipativity. Again, we verified strict dissipativity experimentally by many (10000) uniformly random chosen starting points for the optimization problem (4.5). Therefore we used the Matlab Global Optimization Toolbox, using a local, primal dual optimization algorithm with several restarts. For each run,

[2] We recommend the interested reader to have a closer look at the implementation.

we obtained the same optimizer and therefore we conclude (experimental) strict dissipativity by Thm. 2.3.5 and the discussion on the truck position orbit w.r.t. (5.7).

5.4 Nominal economic model predictive control

In this section, the economic model predictive control scheme for periodic operation (Sec. 3), using a terminal set and a terminal cost is compared against the unconstrained[3] scheme from [18].

Economic MPC with terminal set and cost

For the supply chain example (Fig. 5.1) we use the following terminal set

$$\bar{\mathbb{X}}_f = \tilde{\Pi}_{\mathbb{X}} \tag{5.12}$$

with $\tilde{\Pi}_{\mathbb{X}}$ the set of shifted optimal orbits (2.1), see Def. 1.2.5. As a terminal controller we use the optimal periodic orbit input trajectory $\Pi_{\mathbb{U}}$, starting from the corresponding element with respect to the truck position. It follows that (3.2.1), 3 is fulfilled with equality, independent of the choice of $\bar{V}_f$.

In Fig. 5.2 we show a closed loop trajectory without any unknown disturbances and a planning horizon $N = 8$. Given that special initial condition, an interesting effect is, that some of the goods of the second retail store are distributed to the third retail store. This is because, the second retail store has too many goods in storage with respect to the nominal demand rate by customers. Due to the limitation of the trucks storage, the first retail store needs two periods, before converging to the optimal periodic.

Fig. 5.3 shows the closed loop trajectory, starting with the same configuration but with unknown external disturbances (5.3). Again, as in the simple supply chain example, the retail stores experience a higher demand of goods by the customers compared to the number of available goods in the storage.

[3] In literature, MPC without terminal cost and terminal constraints is often referred to as unconstraint MPC.

Economic MPC without terminal set and cost

Strictly speaking, in order to apply Alg. 2 we must verify the tedious controllability assumptions Ass. 3.6.1 and Ass. 3.6.2. However, due to the increased size of the corresponding P-step system (24 state variables and 8 inputs, when assuming an optimal truck position trajectory), it is no longer possible to use the Multi parametric toolbox [12] for reachability analysis around the optimal orbit because of numerical problems. As a conjecture we think, that the controllability assumptions are still fulfilled. Using an appropriate planning horizon, the results which we introduce later also suggest that this conjecture is true, while the numerical or strict verification is left as an open problem.

The closed loop application of Alg. 2 is shown in Fig. 5.4. We need twice as much time for convergence within the same planning horizon. As in the case of the simple supply chain example, we do not reach the optimal orbit for the specific initial condition at hand. The third retail store has a constant offset of '-1' number of goods compared to the optimal orbit. Since we have to solve a mixed integer problem, it was not possible to increase the planning horizon in case of Alg. 2 further in order to investigate convergence to the optimal periodic orbit in terms of the planning horizon.

In Fig. 5.5 a sample of the closed loop application of Alg. 2 under the presence of unknown, additional disturbances is illustrated.

Comparison

We compare both, Alg. 1 and Alg. 2 also under the presence of disturbances in Tab. 5.1. In contrast to the case of the simple supply chain model (Sec. 1.3), there is a significant performance difference. Using the same planning horizon, our proposed method outperforms the unconstraint strategy. Using shorter planning horizons, as expected, the unconstraint algorithms performance gets worse but remains stable. Importantly, note that due to computation times, we could not apply larger planning horizons than $N = 8$ because of the exponentially increasing computation time of the corresponding mixed integer problem.

5.5 Robust economic model predictive control

Similar to the nominal case, we choose the terminal set

$$\bar{\mathbb{X}}_f = \tilde{\Pi}_{\mathbb{X}}, \tag{5.13}$$

but with (5.11) instead of (5.7). We tighten the state constraints according to (5.10) [4]. Using the auxiliary control law (5.8) and the induced RCI set (5.9) we implement the corresponding integrated stage cost function.

In Fig. 5.6 and Fig. 5.7 are samples of the closed loop behavior using Alg. 3 and Alg. 4 respectively. The average performance is compared in Tab. 5.2. Most importantly, note the instability of Alg. 4. This is due to a randomly different initial condition compared to the previous section and the tightened constraint set. This is a notable drawback of the MPC scheme without terminal constraints [18] because, we will never know for sure, for which planning horizon we can guarantee asymptotic stability. However, since we decided to fix the trucks position trajectory to the optimal orbit we have a convex optimization problem instead of the mixed integer problem in the previous sections. Therefore we can increase the planning horizon. For $N = 10$ we get similar performance between Alg. 3 and Alg. 4, while having comparable computation times. We can conclude, that if we choose the planning horizon appropriately, we get similar performance in the robust setting, due to a simplified problem structure by fixing the trucks trajectory.

Furthermore, note that we have at least halved the average closed loop cost using the robust MPC algorithms, proposed in this work. This essential performance improvement can be pulled down to the concept of the integrated

[4] The trucks position is restricted to the optimal period orbit.

Table 5.1: Comparison of Alg. 1 and Alg. 2 applied to the supply chain model (Fig. 5.1) under disturbances after 400 simulation steps.

Algorithm	Planning hor. N	Avg. Performance
Alg. 1	8	62.98
Alg. 2	6	124.50
Alg. 2	7	102.48
Alg. 2	8	84.46

stage cost function, which we adapted from [4]. The retail store's storage is not at the nominal optimum, but slightly above in order to enable the compensation of unknown customer demands.

Table 5.2: Comparison of Alg. 3 and Alg. 4 applied to the supply chain model (Fig. 5.1) under disturbances after 400 simulation steps.

Algorithm	Planning hor. N	Avg. Performance
Alg. 3	8	31.48
Alg. 4	8	$\to \infty$
Alg. 4	9	40.27
Alg. 4	10	33.35

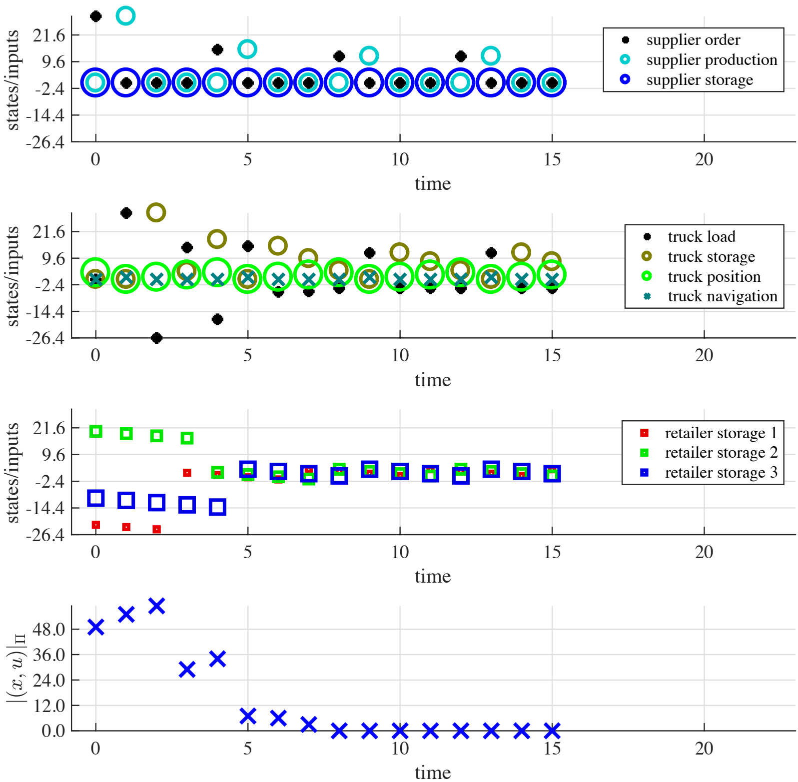

Figure 5.2: Closed loop results of the simple supply chain network using Alg. 1 and planning horizon $N = 8$ without unknown disturbances.

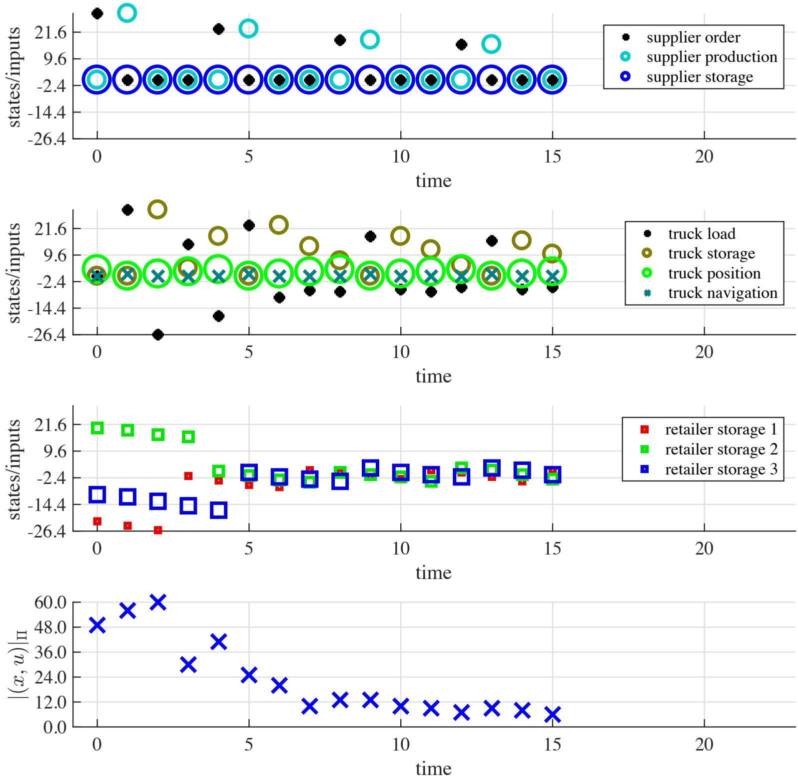

Figure 5.3: Closed loop results of the simple supply chain network using Alg. 1 and planning horizon $N = 8$ with unknown disturbances.

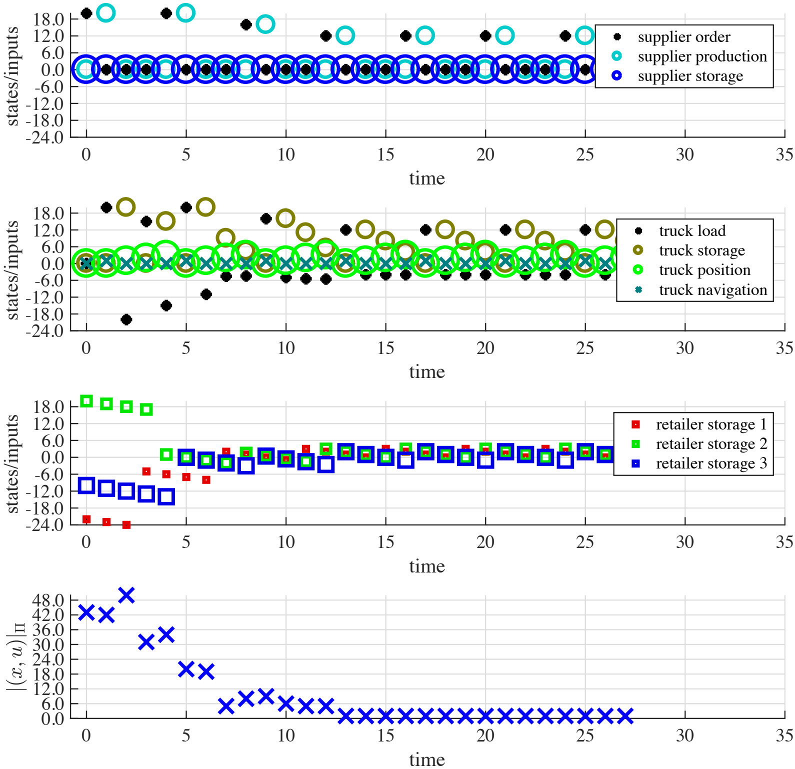

Figure 5.4: Closed loop results of the simple supply chain network using Alg. 2 and planning horizon $N = 8$ without unknown disturbances.

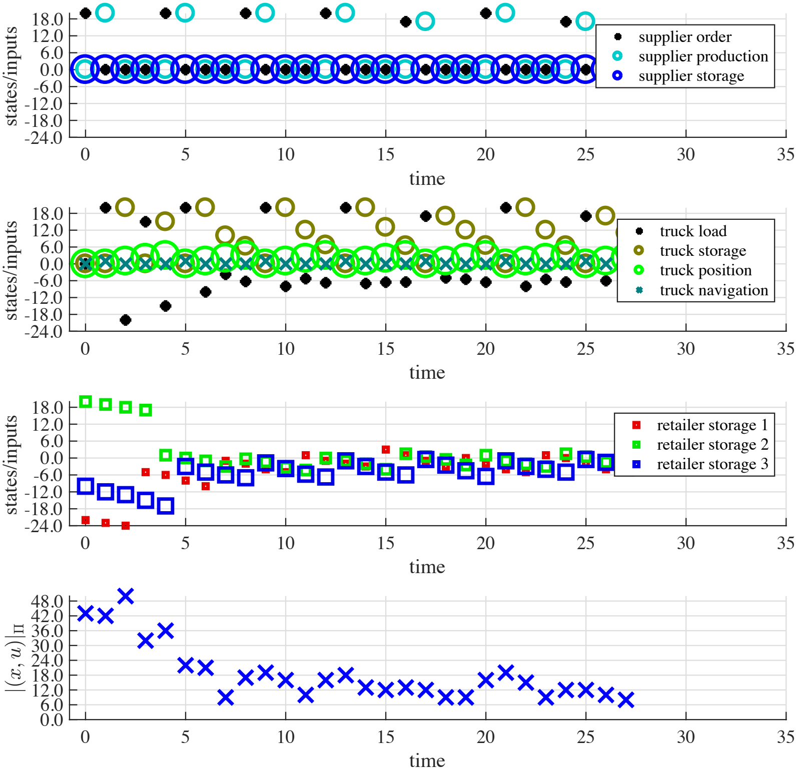

Figure 5.5: Closed loop results of the simple supply chain network using Alg. 2 and planning horizon $N = 8$ with unknown disturbances.

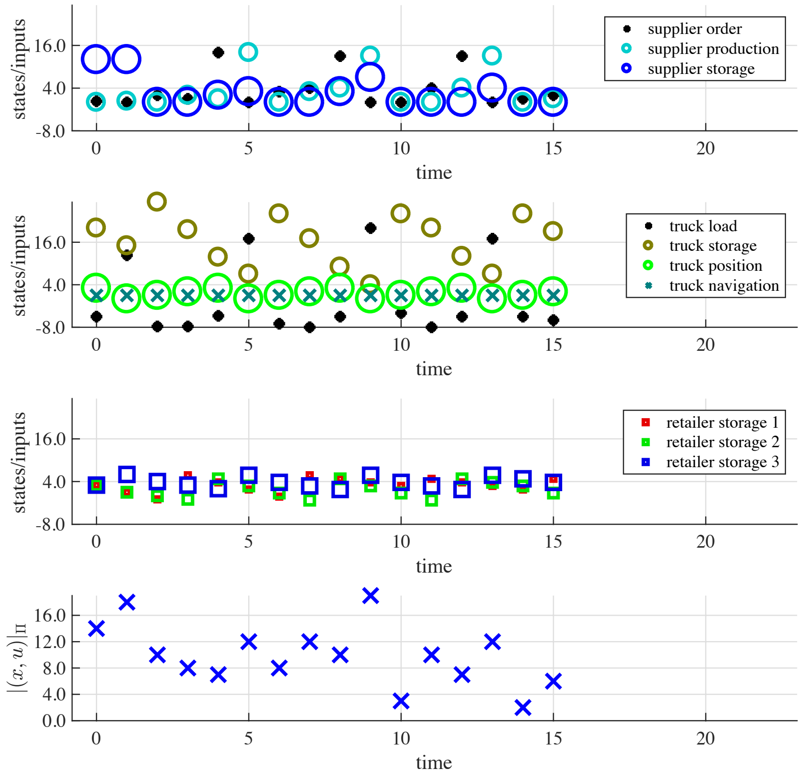

Figure 5.6: Closed loop sample of the simple supply chain network using Alg. 3 and planning horizon $N = 8$ with unknown disturbances.

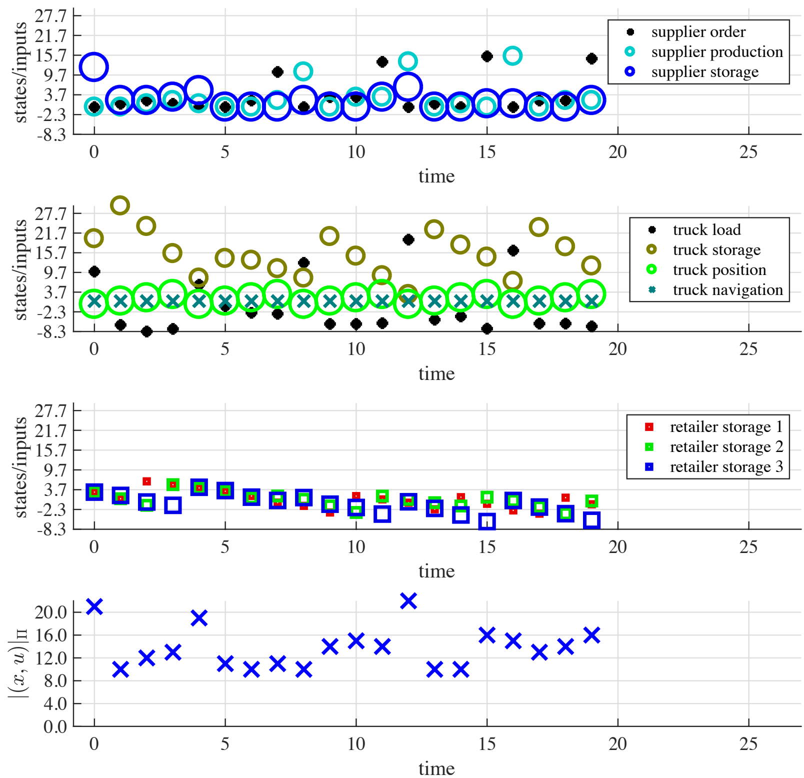

Figure 5.7: Closed loop sample of the simple supply chain network using Alg. 4 and planning horizon $N = 8$ with unknown disturbances.

6 Conclusion

In this work we studied

1. An explicit linear programming formulation in order to verify strict dissipativity with respect to a given periodic orbit in case of linear time varying systems with piece-wise convex cost. In the convex case we further established the necessary and sufficient relation between uniqueness of an optimal periodic orbit and strict dissipativity.

2. A novel economic model predictive control scheme for optimal periodic operation using a terminal region and terminal cost functional. We leveraged existing results from the steady state case to enable a more intuitive (and possibly easier) control design process compared to existing literature. We strictly proved interesting properties, namely recursive feasibility, asymptotic average performance that is no worse than that of the systems optimal periodic orbit, and asymptotic stability of the systems optimal periodic orbit.

3. A robust economic model predictive control scheme in case periodic operation is optimal. To this end we used the concept of integrated stage cost functionals [4].

We demonstrated the methods by using a supply chain network with graph constraints, resulting in a mixed integer programming problem. The problem type is representative for frequently arising problems in the operational supply chain management literature. By comparing our method conceptually to existing methods it turned out that the design process is less involved as in [24], but more complicated as the terminal constraint free method [18]. Caused by the tedious problem structure, we could guarantee optimal asymptotic average performance and asymptotic stability with respect to the optimal (robust) periodic orbit with our method only.

Erklärung

Ich versichere hiermit, dass ich, Kim Peter Wabersich, die vorliegende Masterarbeit selbstständig angefertigt, keine anderen als die angegebenen Hilfsmittel benutzt und sowohl wörtlich als auch sinngemäß entlehnte Stellen als solche kenntlich gemacht habe. Die Arbeit hat in gleicher oder ähnlicher Form noch keiner anderen Prüfungsbehörde vorgelegen.

Ort, Datum

Unterschrift

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